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Dynamic Pricing with Reference Price Dependence

Régis Chenavaz

Abstract

A firm that accounts for consumer behavior sets the selling price of a product considering the reference price of consumers. In the literature, a reference price is usually modeled as depending on past selling prices. That is, past selling prices implicitly constrain the current selling price of a product. In this article, the author explicitly measures this constraint with an optimal control framework. He works on the structural properties of a general demand function, which depends on both selling and reference prices. Analytical results prove the following claims. Adjusting reference prices effects increase the price elasticity of demand, the demand function becoming flatter. Thus, the reference price effect weakens the market power of the firm. Also, the reference price effect constitutes a main driver of the dynamics of the selling price. But contrary to intuition, selling price dynamics does not systematically imitate reference price dynamics.

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Keywords Dynamic pricing; behavioral pricing; reference price dependence; optimal control

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1 Introduction

Standard economics, investigating optimal pricing strategies, assumes a rational consumer for whom the selling price is the sole relevant variable related to price. Behavioral economics, understanding descriptive consumer elements, also integrates a reference price in the decision process of the consumer. A reference price is a personal benchmark against which the customer compares a selling price; a selling price above the reference price looks large and reduces demand, whereas a selling price below the reference price seems low and stimulates demand (Sorger, 1988; Putler, 1992; Kopalle et al., 1996; Lowengart, 2002; Fibich et al., 2003; Paraschiv and Chenavaz, 2011; Zhang et al., 2013, 2014; Li et al., 2015; Lu et al., 2016; Xue et al., 2016). This paper accounts for consumer behavior by integrating reference prices, and it analyzes the dynamic pricing policy in this context.

In this article, I study the determinants of a dynamic pricing policy for a monopolistic firm, when descriptive aspects of consumer behavior are considered. The analysis integrates the main behavioral element of decision making, namely reference price dependence. That is, the demand depends on selling price and reference price. The reference price, a psychological variable internal to the consumer, is formally operationalized by the past observed prices. The literature studying the behavioral element as a driver of dynamic pricing thus informs this research.

This article belongs to the formal behavioral literature on dynamic pricing in which demand evolves adaptively on the basis of the firm's past prices (see the review of Chenavaz et al. 2011). The first attempt to formalize reference price effects originates in Sorger (1988) and is followed by Kopalle et al. (1996). Fibich et al. (2003) show the advantages of continuous time formulation of reference effects. Popescu and Wu (2007) provide the first analysis with a general demand function, and establish structural results. A common point of the aforementioned research is the characterization of the intertemporal equilibrium of the selling price, which is of theoretical interest since product life cycles are supposed infinite or relatively long.

Of practical interest though, the life cycle of most products is finite and relatively short. Thus within a few years period, the intertemporal equilibrium is unlikely to arise in a managerial situation, but the dynamics of price plays a ma-

role worth studying through the optimal path. In contrast to previous research mainly focusing on the intertemporal equilibrium, this article is primarily interested in characterizing the explicit dynamics of the selling price along the optimal path. In this article and following the seminal method of Popescu and Wu (2007), I use a general demand function to establish results linked to the sole properties of the demand function, and independent from any parameter specification. Further and in the vein of the pioneering approach of Fibich et al. (2003), I take advantage of continuous time formulation. Based on optimal control, the simpler modeling enables the characterization of qualitative properties of the optimal path of pricing policy.

In this article, the results support three claims. First, if customers are sensitive to a reference price, then the price elasticity of demand increases (the demand function becomes flatter). In other words, if reference exerts an influence, then the firm extracts lower rents from consumers (its market power decreases). Second, selling price dynamics is decomposed between four competing effects. More precisely, two opposing effects are linked to reference price dynamics and two contradictory effects are tied to anchoring adjustment. Third, the dynamics of price are not systematically associated to the dynamics of the reference price. I provide the analytic conditions of association between selling and reference prices. By integrating descriptive aspects of customer behavior, this paper offers a better understanding of a successful firm pricing policy. A firm ignoring behavioral implications of its pricing policy would charge inadequately for its products, losing profit.

2 Modeling Framework

The firm is in a monopolistic situation. The horizon of the firm T is finite and the time $t \in [0, T]$ is continuous. I describe here how consumers decide to purchase a product on the basis of selling price and reference price.

2.1 Reference Price

A reference price $r(t)$ is an anchor (or benchmark) against which customers compare the current selling price $p(t)$ (Kalyanaram and Winer, 1995; Mazumdar et al., 2005). In most research, consumers build the current reference price as the continuous weighted average of the past selling prices (Winer, 1986; Sorger, 1988; Lowengart, 2002; Fibich et al., 2003; Popescu and Wu, 2007; Afaki and Popescu, 2013; Zhang et al., 2014; Li et al., 2015; Lu et al., 2016; Xue et al., 2016). For an exponentially decaying function with β being the continuous *speed adjustment* parameter (also known as the forgetting rate or memory parameter) and r_0 being the initial reference price at $t = 0$, we have

$$r(t) = e^{-\beta t} \left(r_0 + \beta \int_0^t e^{\beta s} p(s) ds \right), \quad \beta, t \geq 0. \quad (1)$$

Differentiate (1) with respect to time t yields the dynamic of the reference price:

$$\frac{dr(t)}{dt} = \beta(p(t) - r(t)), \quad (2)$$

with the initial reference price condition $r(0) = r_0$.

Equation (2) states that the impact of the price $p(t)$ on the adaptation of the reference price $r(t)$ increases with the adjustment parameter β . In the singular case $\beta = 0$, the price does not affect the reference price adaptation, and the reference price remains constant.

2.2 Demand Formulation

Part of the existing literature that formally models reference effects (Sorger, 1988; Kopalle et al., 1996; Fibich et al., 2003; Zhang et al., 2013, 2014; Xue et al., 2016) considers a linear demand function of the price and the reference price such as

$$D = a - \delta p(t) - \gamma(p(t) - r(t)), \quad a, \delta, \gamma > 0, \quad (3)$$

where a defines the market potential, δ is the marginal impact of price on demand, and γ measures the reference effect. Note that this demand function implies that the consumer does not anticipate future prices.

In this paper, in the vein of Popescu and Wu (2007), Nasiry and Popescu (2011), and Aflaki and Popescu (2013) I generalize the usual linear demand function (3). I study a general reference-dependent demand function $D > 0$, which accounts for nonlinearities and dynamics in response to variations in the reference price:

$$D = D(p(t), r(t)). \quad (4)$$

For tractability, the function D is assumed twice continuously differentiable as in Chenavaz (2011, 2012). The demand decreases (strictly) with price and increases (weakly) with reference price. Moreover, demand decreases (weakly) with price even more with a higher reference price. This assumption of submodularity means that it is more difficult to increase demand by lowering the selling price when the reference price is high than when it is low. Assumptions of strict and weak impacts are made for technical convenience without loss of generality. Where there is no confusion, I omit now the function parameters to simplify the presentation. Formally the conditions write

$$\frac{\partial D}{\partial p} < 0, \frac{\partial D}{\partial r} \geq 0, \frac{\partial^2 D}{\partial p \partial r} \leq 0. \quad (5)$$

The general demand (4) together with the conditions (5) place little restriction on the way (selling) price and reference price affect demand. Indeed, the price effect $\partial D / \partial p < 0$ and the reference effect $\partial D / \partial r \geq 0$ are in line with the usual linear demand (3). Further, the linear demand (3) is enriched by the cross effect $\partial^2 D / \partial p \partial r \leq 0$. Note that the cross effect is null for the class of demand function additively separable $D = h(p) + l(r)$, but it is active for the class of demand function multiplicatively separable $D = h(p)l(r)$, as for say $D = (a - \delta p)e^{-\gamma(p-r)}$. Other examples of multiplicatively separable demand functions include the isoelastic demand function $D = ap^{-\delta}(p-r)^{-\gamma}$ and the exponential demand function $D = ae^{-\delta p}e^{-\gamma(p-r)}$. These functions obviously verify conditions (5). Eventually, the cross effect¹ assumed in (5) may explain a counterintuitive phenomenon, such as a negative selling-reference price linkage.

¹ A similar assumption is discussed in Chenavaz (2016).

3 Dynamic Pricing

I model a monopolist firm in an optimal control framework. Table 1 gives the notations used throughout the article.

Table 1: Notation

T	= fixed terminal time of the planning horizon,
ρ	= interest rate,
$p(t)$	= selling price at time t (decision variable),
$r(t)$	= reference price at time t (state variable),
β	= adjustment speed of the reference price,
$dr(t)/dt$	= $\beta(p - r)$ = reference price dynamics at time t ,
$\lambda(t)$	= current-value adjoint variable at time t ,
$D(p, r)$	= current demand,
$\pi(p, r)$	= $pD(p, r)$ = current profit,
$H(p, r, \lambda)$	= current-value Hamiltonian.

The current profit $\pi > 0$ function is

$$\pi = p(t)D(p(t), r(t)). \quad (6)$$

In line with Popescu and Wu (2007) and Nasiry and Popescu (2011), (6) assumes away the production cost for simplicity, but all the results hold with positive production cost. Because I seek interior solutions (assuming that they exist), the function π is assumed strictly concave in p . This assumption is common in research using a general demand function (Popescu and Wu, 2007; Nasiry and Popescu, 2011; Aflaki and Popescu, 2013).

The firm maximizes the discounted profit by finding the optimal pricing strategy that accounts for the reference price dynamics. With the discount rate $\rho \geq 0$, the problem of the firm is

$$\begin{aligned} & \max_{p(s) \geq 0, s \in [0, T]} \int_0^T e^{-\rho t} \pi(t) dt, \\ & \text{subject to } \frac{dr(t)}{dt} = \beta(p(t) - r(t)), \text{ with } r(0) = r_0. \end{aligned}$$

When charging the price p at each time t , the firm trades off current demand with future demand. Indeed, a higher selling price reduces the current demand but it increase the reference price, and thus expands future demand. The intertemporal profit maximization problem is solved with the necessary and sufficient optimality conditions of Pontryagin's maximum principle. On this basis, the current-value Hamiltonian H formed with the shadow price $\lambda(t)$ (or current-value adjoint variable) writes

$$H(p, r, \lambda) = pD(p, r) + \lambda\beta(p - r). \quad (7)$$

The maximum principle imposes the dynamic of λ

$$\begin{aligned} \frac{d\lambda}{dt} &= \rho\lambda - \frac{\partial H}{\partial r}, \\ &= (\rho + \beta)\lambda - p\frac{\partial D}{\partial r}, \end{aligned} \quad (8)$$

with the transversality condition $\lambda(T) = 0$.

The value of $\lambda(t)$, which measures marginal increase in the reference price $r(t)$ at time t , is obtained through integrating (8) with the transversality condition $\lambda(T) = 0$. The integration yields²

$$\lambda(t) = \int_t^T e^{-(\rho+\beta)(s-t)} p \frac{\partial D}{\partial r} ds. \quad (9)$$

From (9) the shadow price λ is positive over the planning horizon. Formally, $\lambda(t) \geq 0, \forall t \in [0, T]$. In addition, the higher the reference effect $\partial D/\partial r$, the larger λ is. But if the reference effect does not play out ($\partial D/\partial r = 0$), then λ is null.

The current value Hamiltonian H obtained in (7) sums the current and future profits; it measures the instantaneous total profit of the firm at any time t . The firm maximizes the intertemporal profit H if and only if the following necessary and sufficient first- and second-order conditions hold

$$\frac{\partial H}{\partial p} = 0 \implies D + p\frac{\partial D}{\partial p} + \beta\lambda = 0, \quad (10a)$$

$$\frac{\partial^2 H}{\partial p^2} < 0 \implies -2\frac{\partial D}{\partial p} - p\frac{\partial^2 D}{\partial p^2} > 0. \quad (10b)$$

² The proof of (9) is in Appendix.

The first-order condition on price (10a) yields a first result.³

Proposition 1. *Price setting is such that*

$$-\frac{\partial D}{\partial p} \frac{p}{D} = 1 + \frac{\beta \lambda}{D},$$

with λ given by (9).

Proof. Divide (10a) by D and rearrange. □

Proposition 1 states that it is optimal for the firm to set a price such that the price elasticity of demand $-(\partial D/\partial p)/(p/D)$ is elastic (recall $\beta, \lambda \geq 0$ and $D > 0$): that is, if the price increases by 1%, the demand decreases by more than 1%. A higher speed of adjustment β and a higher shadow price of reference price λ are associated to a higher price elasticity of demand. If the reference price does not adapt ($\beta = 0$) or the reference price effect is inactive ($\partial D/\partial r = 0$ implies $\lambda = 0$ from (9)), then the price elasticity of demand is unitary.

The managerial implications are straightforward. The firm operating only at a specific time–static case, because it ignores lasting reference price effects, charges such that the elasticity of demand equals one. In contrast, the firm operating over the whole life cycle of a product–dynamic case, by considering lasting reference price effects, charges such that the elasticity of demand is larger than one. The main empirical implication from Proposition 1 is that the reference dynamics ($\beta > 0$) and the reference effect ($\partial D/\partial r > 0$) cause the firm to loose market power ($-(\partial D/\partial p)/(p/D) > 1$). Intuitively, if consumers are sensitive to a reference price, then it is expected that this reference price weakens the market power of the firm. Proposition 1 offers a formal guarantee to this intuition. Further, Proposition 1 quantify how reference effects weaken the firm’s market power.

The first-order condition on price (10a) must hold over the whole planning horizon. The time decomposition of this condition supports a second result.

³ The proof of (10a) is obvious and the proof of (10b) appears in Appendix.

Proposition 2. *Price dynamics are such that*

$$\frac{dp}{dt} \left(\underbrace{-2 \frac{\partial D}{\partial p} - p \frac{\partial^2 D}{\partial p^2}}_{+} \right) = \frac{dr}{dt} \left(\underbrace{\frac{\partial D}{\partial r}}_{+} + \underbrace{p \frac{\partial^2 D}{\partial p \partial r}}_{-} \right) + \beta \left(\underbrace{(\rho + \beta)\lambda}_{+} - \underbrace{p \frac{\partial D}{\partial r}}_{-} \right),$$

where λ is given in (9).

Proof. Differentiate (10a) with respect to time t . The detailed proof is in Appendix. \square

Proposition 2 quantifies the dynamics of selling price p over time. It represents the first decomposition of the effects tied to the structural properties of the demand function affecting price dynamics along the optimal path. Previous research like Popescu and Wu (2007), Nasiry and Popescu (2011), and Aflaki and Popescu (2013), discuss in great detail the intertemporal equilibrium but not the explicit dynamic of the selling price along the optimal path leading to this equilibrium. In practice, however, the selling period is finite and relatively short—some years or months. Therefore, analyzing price dynamics along the optimal path highlights managerial practice.

By clearly decomposing the effects at play, Proposition 2 clarifies price dynamics when customer behavior is considered. The second-order condition (10b) insures that, on the left-hand side of the formula, the second factor $(-2\partial D/\partial p - \partial^2 D/\partial p^2)$ is positive. On the right-hand side, the first term refers to the impact of the reference price dynamics and the second term to the impact of the adjustment speed. Both impacts, resulting from two competing effects, are ambiguous. If the reference price adjusts over time, the dynamics of price is effective and depends on both the reference price dynamics and adjustment speed impacts. The total reference dynamic impact induced by a reference price increase over time is positively related to the direct reference effect $\partial D/\partial r$ and negatively associated to the indirect reference effect $p \partial D^2/(\partial p \partial r)$.⁴ The total adjustment

⁴ Recall that this last negative effect, driving a possible negative selling-reference price relationship, directly comes from the submodularity assumption (5). Submodularity implications have been discussed in Chenavaz (2016).

impact augments with the investment in reference price $(\rho + \beta)\lambda$ and declines with the deterioration of the reference effect $p \partial D / \partial r$.

Proposition 2 stipulates that if the firm discounts profits ($\rho > 0$), it does not continuously set higher prices leading to continuously higher demand. Further, if the firm does not discount profits ($\rho = 0$), the increase in price over time is even lower (by $\beta^2\lambda$). The rationale is that when future profits count less, the firm has less incentives to increase reference price with higher selling price. The proposition further establishes the conditions under which the selling price is positively associated to the reference price. Consequently, the results undermine the idea that the optimal price monotonically decreases with the reference price or with the adjustment speed. Proposition 2 points out drivers to pricing policies in managerial situations. This pricing rule can thus characterize both skimming and penetration pricing policies, and also a mix of these policies yielding U- and inverted U-shaped pricing curves.

Example 1. *Linear selling-reference price demand function with an interacting effect.*

Enrich the classical linear demand function (3) with an interaction between the selling and the reference price:

$$D = a - \delta p(t) - \gamma(p - r) + \theta p(p - r),$$

with $a, \delta, \gamma, \theta > 0$, and where θ is an interacting parameter enabling the cross effect $\partial^2 D / \partial p \partial r < 0$.

The second order condition (10b) requires $\delta + \gamma - \theta(2 + p) > 0$. The shadow price given by (9) writes

$$\lambda(t) = \int_t^T e^{-(\rho+\beta)(s-t)} p(\gamma - \theta p) ds, \quad (11)$$

with $\gamma - \theta p = \frac{\partial D}{\partial r} \geq 0$.

Proposition 1 imposes

$$(\delta + \gamma - 2\theta p)p = a - \delta p - \gamma(p - r) + \theta p(p - r) + \beta\lambda, \quad (12)$$

with λ given by (11).

Proposition 2 dictates

$$2 \frac{dp}{dt} \left(\underbrace{\delta + \gamma - \theta(2+p)}_{+} \right) = \frac{dr}{dt} \left(\underbrace{\gamma + \theta p}_{+} \right) + \beta \left(\underbrace{(\rho + \beta)\lambda}_{+} - \underbrace{p(\gamma - \theta p)}_{-} \right), \quad (13)$$

with λ given by (11).

Example 1 gives a parametric instance of static pricing rule that has to hold at the optimum with (12) and of dynamic pricing rule describing the evolution of the selling price with (13). The dynamic pricing rule verifies that, depending on the value of the parameters, the evolution of the selling price does not have to match the evolution of the reference price; a negative selling-reference price relationship may appear.

4 Discussion and Conclusion

Reference price plays a role observable on the decision of consumers, but it is not that we have practical examples of all theoretical effects, especially for the indirect effects (even though they may play a role). Still, it is interesting from a theoretical point of view to know that these effects can exert influence on the pricing policy. But a possible theoretical effect at play is a sales effect (in line with Chenavaz 2016). Higher reference point increases demand. So the firm may be tempted to increase the selling price. But sales increase with higher reference price even more if the selling price decreases. So this indirect effect of the reference price plays in the opposite direction of the direct effect of the reference price. Therefore, if the indirect effect is larger than the direct effect, then an increase in the reference price implies a reduction of the selling price. Because this indirect effect theoretical and based on a psychological construct, it is hard to find an example. But this does not imply that the effect does not play a role. More importantly, if the firm disregards this counter-intuitive effect, then it loses money. This article provides two main results

A first result, Proposition 1, expresses that if consumers respond to a reference price, then the demand is more elastic. In other words, the monopoly loses some

market power in the sense that it can extract less rent from the demand, and it has to charge less to maximize profits. The reason of loss of market power (that is, of elasticity being more elastic or of demand function becoming flatter) is that past prices play a role in determining current demand. That is the monopoly, while setting prices today, is constraint by past prices that he proposed to the consumer himself. This additional constraint, demand being sensitive to past prices, explains the loss of market power and greater elasticity of demand.

A second result, Proposition 2, formulates that contrary to intuition, a higher reference price for the consumer does not always imply a higher selling price. In other words, reference price dynamics does not imitate selling price dynamics. Instead, the final impact of an increased reference price on the selling price is the sum of four competing effects. The proposition provides the first explicit decomposition of the effects at play along the optimal path. The first two effects are related to reference price dynamics. Thus a higher reference price develops the demand, enabling a higher selling price (positive effect). The demand, however, would be even higher with a lower selling price (negative effect). The last two effects are associated with reference price adjustment. On one side, a faster adjustment encourages the firm to increase the price, thereby augmenting the reference price and developing demand in the future (positive effect). On the other side, a faster adjustment also means lower memory of past prices, and the interest for a higher reference price will decline more quickly (negative effect).

Depending on the sensitivity of consumer to the reference price and on the adjustment possibility of the reference price, three cases arise.

- **No-adjustment baseline case:** $\beta = 0$. If there is no reference price adjustment, then 1) the monopoly enjoys full market power and 2) the price is constant over time. Indeed, the firm prices such that the price elasticity of demand equals one, benefiting from standard market power ($\partial D / \partial p / p / D = 1$ from Proposition 1). Also the reference price and the selling price are constant over the planning horizon ($dr/dt = 0$ from Proposition 2 and $dp/dt = 0$ from Proposition 2).
- **No-reference baseline case:** $\partial D / \partial r = 0$. If there is no reference price effect, then the case is equivalent to the above no-adjustment baseline case. That is, the firm sets a price for which the price elasticity of demand is

unitary and this price is constant over time. If effect, if $\partial D/\partial r = 0$, then $\lambda = 0$ according to (9). Then the price elasticity is unitary from Proposition 1 and the price is constant following Proposition 2. Consequently, the absence of a reference adjustment (case above) or of a reference effect (this case) yield equivalent results.

- **Adjustment and reference case:** $\beta > 0$ and $\partial D/\partial r > 0$. If the reference price 1) adjusts and 2) affects the demand, then the firm loses some market power and changes the price over time. Indeed, the demand is elastic ($\partial D/\partial p/p/D > 1$ from Proposition 1) and price evolves over time (the dynamics of price plays a role following Proposition 2). Interestingly, both reference adjustment and reference effect are conditions for weaker market power and price dynamics of a monopolist. So the monopolist may have an incentive to lower the level of the reference effect and its adjustment speed. Also, when the firm takes into account the impact on the future reference price of its current selling price (which is obvious), it also has to take into account that the trade-off between current selling price and future reference price is actually more tricky than it appears at first glance: indeed, the firm has to recognize that the presence of reference effects undermines its ability to charge at the monopolist level.

The modeling assumes that 1) the firm behaves in a monopoly market and 2) consumers do not anticipate future prices. The introduction of competition within the model would not make the forces pushing toward a negative selling-reference price relationship vanish. But the introduction of future prices anticipation by consumer may alter the results. That is, the qualitative insights of the model would hold with other market conditions, though not necessarily with more sophisticated consumers. Such limitations call, in turn, for further research on the subject.

To conclude, I provide a qualitative analysis of the optimal path of the dynamic pricing policy resulting from a general demand function. This structural approach captures with little restriction consumer behavioral effects related to reference dependence. Any firm that misunderstands these general effects loses profit by charging inadequately for its product. This work provides thus a deeper understanding of dynamic pricing policies when customers are subject to refer-

ence effects. Such theoretical implications, in turn, call for empirical support in future research.

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Appendix

Proof of Equation (9)

Recall that the dynamic of λ writes in (8)

$$\frac{d\lambda(t)}{dt} = (\rho + \beta)\lambda(t) - p\frac{\partial D}{\partial r}, \text{ with } \lambda(T) = 0.$$

Consider the integrating factor $e^{-(\rho+\beta)t}$, such that

$$\frac{d\lambda(t)e^{-(\rho+\beta)t}}{dt} = e^{-(\rho+\beta)t} \left(\frac{d\lambda(t)}{dt} - (\rho + \beta)\lambda(t) \right).$$

Since $\frac{d\lambda(t)}{dt} - (\rho + \beta)\lambda(t) = -p\frac{\partial D}{\partial r}$, then

$$\begin{aligned} \frac{d\lambda(t)e^{-(\rho+\beta)t}}{dt} &= e^{-(\rho+\beta)t} \left(-p\frac{\partial D}{\partial r} \right), \\ \int_t^T d\lambda(s)e^{-(\rho+\beta)s} &= \int_t^T e^{-(\rho+\beta)s} \left(-p\frac{\partial D}{\partial r} \right) ds, \\ \lambda(T)e^{-(\rho+\beta)T} - \lambda(t)e^{-(\rho+\beta)t} &= \int_t^T e^{-(\rho+\beta)s} \left(-p\frac{\partial D}{\partial r} \right) ds. \end{aligned}$$

Substitute the transversality condition $\lambda(T) = 0$ yields

$$\lambda(t) = \int_t^T e^{-(\rho+\beta)(s-t)} \left(p\frac{\partial D}{\partial r} \right) ds,$$

which completes the proof.

Proof of Equation (10b)

The first-order condition with respect to p (10a) writes

$$\frac{\partial H}{\partial p} = 0 \implies D + p \frac{\partial D}{\partial p} + \beta \lambda = 0,$$

The second-order condition with respect to p imposes

$$\begin{aligned} \frac{\partial^2 H}{\partial p^2} < 0 &\implies \frac{\partial D}{\partial p} + \frac{\partial D}{\partial p} + p \frac{\partial^2 D}{\partial p^2} < 0, \\ &\implies -2 \frac{\partial D}{\partial p} - p \frac{\partial^2 D}{\partial p^2} > 0, \end{aligned}$$

which completes the proof.

Proof of Equation (2)

The first-order condition with respect to p (10a) states

$$\frac{\partial H}{\partial p} = 0 \implies D + p \frac{\partial D}{\partial p} + \beta \lambda = 0.$$

Derivate this condition with respect to t :

$$\begin{aligned} \frac{d}{dt} \left(D + p \frac{\partial D}{\partial p} + \beta \lambda \right) &= 0, \\ \frac{\partial D}{\partial p} \frac{dp}{dt} + \frac{\partial D}{\partial r} \frac{dr}{dt} + \frac{dp}{dt} \frac{\partial D}{\partial p} + p \left(\frac{\partial^2 D}{\partial p^2} \frac{dp}{dt} + \frac{\partial^2 D}{\partial p \partial r} \frac{dr}{dt} \right) + \beta \frac{d\lambda}{dt} &= 0. \end{aligned}$$

Substitute the dynamics of λ from (8) gives

$$\begin{aligned} \frac{\partial D}{\partial p} \frac{dp}{dt} + \frac{\partial D}{\partial r} \frac{dr}{dt} + \frac{dp}{dt} \frac{\partial D}{\partial p} + p \left(\frac{\partial^2 D}{\partial p^2} \frac{dp}{dt} + \frac{\partial^2 D}{\partial p \partial r} \frac{dr}{dt} \right) + \beta \left((\rho + \beta) \lambda - p \frac{\partial D}{\partial r} \right) &= 0, \\ \frac{dp}{dt} \left(-2 \frac{\partial D}{\partial p} - p \frac{\partial^2 D}{\partial p^2} \right) = \frac{dr}{dt} \left(\frac{\partial D}{\partial r} + p \frac{\partial^2 D}{\partial p \partial r} \right) + \beta \left((\rho + \beta) \lambda - p \frac{\partial D}{\partial r} \right), \end{aligned}$$

which completes the proof.

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