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Extending Removed Sets Revision to partially preordered belief bases

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Most of belief revision operations have been proposed for totally preordered information. However, in case of partial ignorance, pieces of information are partially preordered and few effective approaches of revision have been proposed. The paper presents a new framework for revising partially preordered information, called Partially Preordered Removed Sets Revision (PPRSR). The notion of removed set, initially defined in the context of the revision of non ordered or totally preordered information is extended to partial preorders. The removed sets are efficiently computed thanks to a suitable encoding of the revision problem into logic programming with answer set semantics. This framework captures the possibilistic revision of partially preordered information and allows for implementing it with ASP. Finally, it shows how PPRSR can be applied to a real application of the VENUS european project.

1. Introduction

Belief revision has been extensively studied in the domain of knowledge representation for artificial intelligence, mainly for totally preordered information. A characterization of belief revision has been provided by Alchourron, Gärdenfors, Makinson (AGM) with a set of postulates that any revision operation should satisfy [22]. Katsuno and Mendelzon (KM) reformulated AGM’s postulates and provided a representation theorem that characterizes revision operations based on total preorders [30]. Belief revision has been discussed within different frameworks (probabillity theory, Sphon’s conditional functions, Grove’s system of spheres, …) and several revision operations have been proposed like the possibilistic revision [20,21] or adjustment revision [46], the linear-based revision [38], the natural revision [12], the lexicographic-based revision [1,37,34], the revision with memory [2,40], and more recently several works on iterated revision [10,28,11,19]. As pointed out by Delgrande et al. [18], the different approaches proposed in the literature can be classified according to three different points of view. Given a plausibility ordering on interpretations describing the background knowledge and a new piece of information, belief revision as defeasible inference (BRDI) amounts to find the most plausible interpretation satisfying the input information, belief revision as incorporation of evidence (BRIE) amounts to change the plausibility ordering in presence of a new piece of information and belief revision of background knowledge (BRBK) means revising the background knowledge by a generic information.

In order to provide effective revision operations, strategies have to be chosen. When a new piece of information is inconsistent with the initial beliefs, the revision problem is related to the problem of consistency restoration. This problem has been addressed according to three main families of approaches: construction of preferred consistent (or maximal consistent)
subsets of formulae \([42,1,15]\), forgetting some variables responsible of the inconsistency \([36,33,9]\), inconsistency minimization \([41,14,25,48]\).

Some approaches have been implemented \([47,17]\), among them, Removed Sets Revision which has been initially proposed in \([48]\) for revising a set of propositional formulae. Contrary to consistency maximization, this approach stems from removing a minimal number of formulae, called removed set, to restore consistency. The Removed Sets Revision (RSR) and then a prioritized form of Removed Sets Revision, called Prioritized Removed Sets Revision (PRSR) \([8]\) have been encoded into answer set programming and allowed for solving a practical revision problem coming from a real application in the framework of geographical information system.

However in some applications, an agent has not always a total preorder between situations at his disposal, but is only able to define a partial preorder between situations, particularly in case of partial ignorance and incomplete information. In such cases, an epistemic state can be represented by either a partial preorder on interpretations or a partially preordered belief base.

The revision of partially preordered information has been less investigated in the literature, however Lagrue and coworkers \([6]\) pointed out that the KM’s postulates are not appropriate for partial preorders and proposed a suitable definition of faithful assignment, called P-faithful assignment, a new set of postulates and a representation theorem. Some revision operations initially defined for total preorders, such as revision with memory and possibilistic revision have been successfully extended to partial preorders \([4]\).

This paper proposes a new framework for revising partially preordered information and provides an efficient implementation thanks to answer set programming. The main contributions of this paper are the following:

- It extends the Removed Sets Revision to partially preordered information, called Partially Preordered Removed Sets Revision (PPRSR). The paper shows how the notion of removed set, roughly speaking, the subsets of formulae to remove to restore consistency, initially defined in the context of non ordered \([48]\) or totally ordered \([8]\) information is extended to the case of the revision of partially preordered information.
- It provides an implementation of PPRSR with ASP. The revision problem is translated into a logic program with answer set semantics and a one-to-one correspondence between removed sets and preferred answer sets is shown. The computation of answer sets is performed with any ASP solver.
- It shows that the possibilistic revision of partially preordered information can be captured within the PPRSR framework allowing for an efficient implementation with ASP.

The rest of this paper is organized as follows. Section 2 fixes the notations and gives a refresher on the Removed Sets Revision (RSR), on answer set programming and on partial preorders. Section 3 presents the Partially Preordered Removed Set Revision (PPRSR) and shows how it captures the possibilistic revision. Section 4 details the encoding of PPRSR into logic programming with answer set semantics and the computation of answer sets thanks to ASP solvers. It then shows the one-to-one correspondence between removed sets and preferred answer sets. Section 5 illustrates how PPRSR can be applied in the context of the VENUS project before concluding.

2. Background and notations

2.1. Notations

In this paper we use propositional calculus, denoted by \(\mathcal{L}_{\text{PC}}\), as knowledge representation language with usual connectives \(\neg, \land, \lor, \rightarrow, \leftrightarrow\). Let \(X\) be a set of propositional formulae, we denote by \(\text{Cons}(X)\) the set of logical consequences of \(X\). We denote by \(\mathcal{W}\) the set of interpretations of \(\mathcal{L}_{\text{PC}}\) and by \(\text{Mod}(\psi)\) the set of models of a formula \(\psi\), that is \(\text{Mod}(\psi) = \{\omega \in \mathcal{W}, \omega \models \psi\}\) where \(\models\) denotes the inference relation used for drawing conclusions.

2.2. Removed Sets Revision

We briefly recall the Removed Sets Revision approach. Removed Sets Revision \([48]\) deals with the revision of a set of propositional formulae by a set of propositional formulae.\(^1\) Let \(K\) and \(A\) be finite sets of clauses. Removed Sets Revision (RSR) focuses on the minimal subsets of clauses to remove from \(K\), called removed sets, in order to restore the consistency of \(K \cup A\). More formally,

**Definition 1.** Let \(K\) and \(A\) be two consistent sets of clauses such that \(K \cup A\) is inconsistent. \(R\) a subset of clauses of \(K\), is a removed set of \(K \cup A\) iff

\[\begin{align*}
(i) \quad & (K \cup A) \cup R \text{ is consistent;} \\
(ii) \quad & \forall R \subseteq K, \text{ if } (K \cup R) \cup A \text{ is consistent then } |R| \leq |R'|.\end{align*}\]

Let denote by \(R(K \cup A)\) the collection of removed sets of \(K \cup A\), RSR is defined as follows:

\(^1\) The initial approach considers propositional formulae in their equivalent conjunctive normal form (CNF).
\(^2\) \(|R|\) denotes the number of clauses of \(R\).
Definition 2. Let $K$ and $A$ be two consistent sets of clauses,
\[
K_{\text{RSR}} A =: \bigvee_{R \in \mathcal{R}(K, A)} \text{Cons}(K \setminus R \cup A).
\]

According to a semantic point of view, $|\mathcal{N}_K(\omega)|$ denotes the number of clauses of $K$ falsified by an interpretation $\omega$ and a total preorder on interpretations is defined by:

Definition 3. $\omega_1 \preceq \omega_2$ iff $|\mathcal{N}_K(\omega_1)| \leq |\mathcal{N}_K(\omega_2)|$.

Removed Sets Revision can be semantically defined by:

Definition 4. Let $K$ and $A$ be two consistent sets of clauses,
\[
\text{Mod}(K_{\text{RSR}} A) = \min(\text{Mod}(A), \leq_K).
\]

It minimizes the number of clauses falsified by the models of $A$ and $\text{Mod}(K_{\text{RSR}} A) = \text{Mod}(K_{\text{RSR}} A)$. In case of prioritized belief bases, RSR has been extended to Prioritized Removed Sets Revision (PRSR) [8].

2.3. Partial preorders

A partial preorder, denoted by $\preceq$ on a set $A$ is a reflexive and transitive binary relation. Let $x$ and $y$ be two members of $A$, the equality is defined by $x = y$ iff $x \preceq y$ and $y \preceq x$. The corresponding strict partial preorder, denoted by $\prec$, is such that, $x \prec y$ iff $x \preceq y$ holds but $x = y$ does not hold. We denote by $\sim$, denoted by $\text{Min}(A, \sim)$, is defined as: $\text{Min}(A, \sim) = \{x \in A, \exists y \in A: y \prec x\}$. An epistemic state allows for encoding the agent’s beliefs but also for encoding its relative confidence in alternative possible states of the world. Epistemic states can be represented by several means: preorders on interpretations [16,34], conditionals [13,16], epistemic entrenchments [45,37], prioritized belief bases [2,3], … When focusing on representations by means of preorders, epistemic states are generally represented by total preorder on interpretations, however, as mentioned in the introduction, in case of partial ignorance, the agent is unable to compare all situations between them and a partial preorder seems to be more suitable to represent epistemic states.

Katsuno and Mendelzon [30] proposed a set of postulates and a representation theorem that characterize revision operations based on partial preorders. However, the proposed approach is not satisfactory since only one class of partial preorders can be revised [6]. In particular, the concept of faithfull assignment defined in [30] is unable to represent all partial preorders on interpretations. Let $\Psi$ be an epistemic state represented by a partial preorder on interpretations, denoted by $\preceq$ and let $\text{Bel}(\Psi)$ be its corresponding belief set. An alternative definition of faithfull assignment, called P-faithful assignment, is proposed in [6] as follows.

Definition 5. Let $\text{Bel}(\Psi) = \min(\mathcal{W}, \prec_\Psi)$, $\preceq_\Psi$ is a P-faithful assignment if

1. if $\omega, \omega' \Vdash \text{Bel}(\Psi)$ then $\omega \prec_\Psi \omega'$ does not hold,
2. if $\omega' \nVdash \text{Bel}(\Psi)$, then there exists $\omega$ such that $\omega \Vdash \text{Bel}(\Psi)$ and $\omega \prec_\Psi \omega'$,
3. if $\Psi = \Phi$ then $\preceq_\Psi = \preceq_\Phi$.

Moreover, [6] gives a set of postulates an operation $\circ$ has to satisfy and a representation theorem such that $\text{Mod}(\text{Bel}(\Psi \circ \mu)) = \min(\text{Mod}(\mu), \leq_\Psi)$. An alternative syntactic but equivalent representation of an epistemic state, $\Psi$ is a partially preordered belief base, denoted by $(\Sigma, \preceq, \Sigma)$, where $\Sigma$ is a set of propositional formulae, and $\leq_\Sigma$ is a partial preorder on the formulæ of $\Sigma$. Several ways of defining a partial preorder on subsets of formulæ belonging to $\Sigma$, called comparators, from a partial preorder on a set of formulæ $\Sigma$ have been proposed: inclusion-based [29], possibilistic [5], lexicographic [49] comparators. They are such that the preferred formulæ are kept in the belief base. In our approach, according to the Removed Sets strategy, we adopt a dual point of view in the sense that we want to prefer the subsets of formulæ to remove. For example, we rephrase the possibilistic comparator (or weak comparator) used in [5], already defined in [35] and reused by Halpern [26] as follows. $Y$ is preferred to $X$ if for each element of $Y$, there exists at least one element of $X$ which is preferred to it, more formally:

Definition 6. Let $\preceq_\Sigma$ be a partial preorder on $\Sigma$, $Y \subseteq_\Sigma \Sigma$ and $X \subseteq_\Sigma \Sigma$.

$Y$ is preferred to $X$, denoted by $Y\preceq_\Sigma X$ iff $\forall y \in Y, \exists x \in X$ such that $x \preceq_\Sigma y$.

We now briefly recall the extension of the semantic possibilistic revision to partial preorders [4]. The semantics of possibilistic logic stems from the notion of possibility distribution [20], denoted by $\pi$, which is a function from $\mathcal{W}$ to $[0, 1]$. $\pi(\omega)$ evaluates to what extent $\omega$ is compatible, or consistent with the available knowledge. $\pi(\omega) = 0$ means that $\omega$ is impossible, while $\pi(\omega) = 1$ means that $\omega$ is totally possible. $\pi(\omega) \preceq \pi(\omega')$ means that $\omega$ is more plausible than $\omega'$. A possibility distribution is said to be normalized or consistent if there exists an interpretation $\omega$ such that $\pi(\omega) = 1$.

An epistemic state $\Psi$ is represented by $(\mathcal{W}, \preceq_\Psi)$ where the partial preorder on interpretations $\preceq_\Psi$ is associated to a possibility distribution as follows: $\forall \omega, \omega' \in \mathcal{W}, \omega \preceq_\Psi \omega' \text{ iff } \pi(\omega) \preceq \pi(\omega')$. The possibilistic revision of $\Psi$ by a propositional
formula \( \mu \) leads to the epistemic state \( \Psi_{\circ \tau \mu \circ} \) represented by \((W, \preceq_{\Psi_{\circ \tau \mu \circ}})\) which considers all the counter-models of \( \mu \) as impossible and preserves the relative ordering between the models of \( \mu \). More formally,

**Definition 7.** \( \Psi_{\circ \tau \mu} \) corresponds to the following partial preorder:

1. if \( \omega, \omega' \in \text{Mod}(\mu) \) then \( \omega \preceq_{\Psi_{\circ \tau \mu}} \omega' \) iff \( \omega \preceq_{\Psi} \omega' \),
2. if \( \omega \not\in \text{Mod}(\mu) \) then \( \omega =_{\Psi_{\circ \tau \mu}} \omega' \),
3. if \( \omega \in \text{Mod}(\mu) \) and \( \omega' \not\in \text{Mod}(\mu) \) then \( \omega \prec_{\Psi_{\circ \tau \mu}} \omega' \).

2.4. Answer sets

A normal logic program is a set of rules of the form \( c \leftarrow a_1, \ldots, a_n, \neg b_1, \ldots, \neg b_m \) where \( c, a_i(1 \leq i \leq n), b_j(1 \leq j \leq m) \) are propositional atoms and the symbol \( \not \) stands for negation as failure. For a rule \( r \) like above, we introduce head\( (r) = c \) and body\( (r) = \{a_1, \ldots, a_n, \neg b_1, \ldots, \neg b_m\} \). Furthermore, let body\({}^+ (r) = \{a_1, \ldots, a_n\} \) denotes the set of positive body atoms and body\({}^- (r) = \{b_1, \ldots, b_m\} \) the set of negative body atoms, and body\( (r) = \text{body}^+(r) \cup \text{body}^-(r) \). Let \( r \) be a rule, \( r^* \) denotes the rule head\( (r) \leftarrow \text{body}^+(r) \), obtained from \( r \) by deleting all negative body atoms in the body of \( r \).

A set of atoms \( X \) is closed under a basic program \( P \) iff for any rule \( r \in P \), head\( (r) \in X \) whenever body\( (r) \subseteq X \). The smallest set of atoms which is closed under a basic program \( P \) is denoted by \( \text{CN}(P) \). The reduct or Gelfond–Lifschitz transformation [24], \( P^X \) of a program \( P \) relatively to a set \( X \) of atoms is defined by \( P^X = \{r^* | r \in \text{Pand body}^-(r) \cap X = \emptyset\} \). A set of atoms \( X \) is an answer set of \( P \) iff \( \text{CN}(P^X) = X \).

3. Partially Preordered Removed Sets Revision (PPRSR)

Let \( \Psi \) be an epistemic state for partially preordered information and \( \text{Bel}(\Psi') \) its corresponding belief set. \( \Psi' \) can be interpreted according either a syntactic or a semantic point of view. \( \Psi' \) is syntactically represented by \((\Sigma, \preceq_\Sigma)\) where \( \Sigma \) is a finite set of arbitrary formulae and \( \preceq_\Sigma \) is a partial preorder on \( \Sigma \). Moreover, \( \Psi' \) can also be semantically represented as a partial preorder on interpretations such that the models of \( \text{Bel}(\Psi') \) are minimal with respect to this partial preorder.

3.1. The PPRS framework

Revising partially preordered belief bases involves the definition of a partial preorder on subsets of formulae, called comparators [5,49]. We first present a general framework, Partially Preordered Removed Sets Revision (PPRSR) without specifying a particular comparator. We start with the syntactic approach then with the semantic one for revising an epistemic state \( \Psi \) by a formula \( \mu \).

**PPRSR Syntactic Approach.** The original Removed Sets Revision was proposed for revising belief bases consisting of propositional formulae in conjunctive normal form, we now extend Removed Sets Revision to arbitrary formulae. Obviously, the Removed Sets Revision of arbitrary formulae does not lead to the same results than the Removed Sets Revision of formulae in CNF. However, only considering formulae in CNF, we loose the syntactic structure of the initial beliefs. Moreover, in some applications, it does not make sense to only remove part of formulae to restore consistency. In [32] the approach concentrated to the construction of maximal consistent subbases from the initial belief base while our strategy is to focus on subsets of formulae to remove to restore consistency. More precisely, our strategy is to determine the preferred subsets (with respect to the initial preorder on formulae) to remove, called removed sets.

According to the syntactic point of view, the revision of the epistemic state \( \Psi \) by a formula \( \mu \) leads to an epistemic state denoted by \( \Psi_{\circ \tau \mu} \) and represented by a new partially preordered belief base \((\Sigma_{\circ \tau \mu}, \preceq_{\Sigma_{\circ \tau \mu}})\) where the partial preorder \( \preceq_{\Sigma_{\circ \tau \mu}} \) is such that \( \mu \) is preferred to any formula of \( \Sigma \) and the relative ordering between the formulae of \( \Sigma \) is preserved. More formally,

**Definition 8.** Let \((\Sigma, \preceq_\Sigma)\) be a syntactic representation of \( \Psi \), the revision of \( \Psi \) by a formula \( \mu \) leads to the revised epistemic state denoted by \( \Psi_{\circ \tau \mu} \) represented by a partially preordered belief base \((\Sigma_{\circ \tau \mu}, \preceq_{\Sigma_{\circ \tau \mu}})\) where

- \( \Sigma_{\circ \tau \mu} = \Sigma \cup \{\mu\} \),
- \( \preceq_{\Sigma_{\circ \tau \mu}} : (i) \forall \psi \in \Sigma, \mu \prec_{\Sigma_{\circ \tau \mu}} \psi \) and \( (ii) \forall \phi \in \Sigma, \psi \preceq_{\Sigma_{\circ \tau \mu}} \phi \) iff \( \psi \preceq_\Sigma \phi \).

Since \( \Sigma \cup \{\mu\} \) may be inconsistent, we have to provide the consistent belief set, denoted by \( \text{Bel}(\Psi_{\circ \tau \mu}) \), corresponding to the revised epistemic state. In order to syntactically compute \( \text{Bel}(\Psi_{\circ \tau \mu}) \) we focus on the preferred subsets of formulae, with respect to the initial partial preorder, to remove from \( \Sigma \), in order to restore consistency. We first define the potential removed sets as follows:

**Definition 9.** Let \((\Sigma, \preceq_\Sigma)\) be a syntactic representation of \( \Psi \). Let \( \mu \) be a formula such that \( \Sigma \cup \{\mu\} \) is inconsistent. \( R \), a subset of formulae of \( \Sigma \), is a potential removed set of \( \Sigma \cup \{\mu\} \) iff \( \Sigma \cup \{\mu\} \) is consistent.
allow us to remove the formulae that are not preferred according to Definition 11. Let $(a, b, a \lor \neg b, \neg a \lor b)$ be a given partial preorder illustrated by Fig. 1 where $b \prec a$ means that $b <_{\Sigma} a$. We revise $\Sigma$ by $\mu = \neg a \lor b$. The potential removed sets are $R_0 = \{a, a \lor \neg b\}$, $R_1 = \{a, a \lor \neg b, \neg a \lor b\}$, $R_2 = \{a, b, a \lor \neg b, \neg a \lor b\}$, $R_3 = \{b, \neg a \lor b\}$, $R_4 = \{b, a \lor \neg b, \neg a \lor b\}$, $R_5 = \{b, a \lor \neg b, \neg a \lor b\}$. $R_6 = \{a, b, \neg a \lor b\}, R_7 = \{a, b\}.

Let $R(\Sigma, \mu)$ be the set of potential removed sets. Among them, we want to prefer the potential removed sets which allow us to remove the formulae that are not preferred according to $\preceq_{\Sigma}$. This leads to define a partial preorder on subsets of formulae of $\Sigma$, called comparator $[5, 49]$, denoted by $\preceq_{C}$. We now generalize the notion of removed sets to subsets of partially preordered formulae. We denote by $R_C(\Sigma, \mu)$ the set of removed sets of $\Sigma, \mu$.

**Definition 10.** Let $(\Sigma, \preceq_{\Sigma})$ be a syntactic representation of $\Psi$. Let $\mu$ be a formula such that $\Sigma, \mu$ is inconsistent. $R \subseteq \Sigma$ is a removed set of $\Sigma, \mu$ iff

1. $R$ is a potential removed set.
2. $R' \in R(\Sigma, \mu)$ such that $R' \supseteq R$.
3. $R' \in R(\Sigma, \mu)$ such that $R' \supseteq R$.

**Example 2.** In the examples, we will use the weak comparator, denoted by $\preceq_{w}$ and defined in Section 2.3. We have $R_0 \preceq_{w} R_1$ because $a \preceq_{w} a$ and $\neg a \lor b \preceq_{w} a \lor \neg b$. The partial preorder on the potential removed sets is illustrated in Fig. 2.

**Remark.** We could refine the notion of removed set with an extra preference according to a strategy $P$ (cardinality or minimality). $R_{C,P}(\Sigma, \mu)$ denotes the set of removed sets of $\Sigma, \mu$ according to the strategy $P$. In this case, a preferred removed set according to a strategy $P$ is a removed set $R$ such that $R' \in R_{C,P}(\Sigma, \mu)$ such that $R' \supseteq R$. According to the cardinality, $R_{V} \preceq_{\text{CARD}} R_{X}$ iff $|V| \subseteq |X|$ with $|X|$ the cardinality of the set $X$. According to the minimality, $R_{V} \preceq_{\text{MIN}} R_{X}$ iff $|V \cap \text{MIN} \subseteq |X \cap \text{MIN}|$, with $\text{MIN} = \{x \in \Sigma, y \in \Sigma, y <_{\Sigma} x\}$.

**Example 3.** We can apply strategies: $R_{w,\text{CARD}}(\Sigma, \mu) = \{R_0, R_1\}$ and $R_{w,\text{MIN}}(\Sigma, \mu) = \{R_0\}$.

Stemming from the above notion of removed set, the belief set corresponding to the revised epistemic state, denoted by $\text{Bel}(\Psi \circ_{\Sigma} \mu), \text{is defined as follows:}$

**Definition 11.** Let $(\Sigma, \preceq_{\Sigma})$ be the syntactic representation of $\Psi$ and let $\mu$ be a formula, the belief set corresponding to the revised epistemic state $\Psi \circ_{\Sigma} \mu$ is $\text{Bel}(\Psi \circ_{\Sigma} \mu) = \bigvee_{R \in R_{C}(\Sigma, \mu)} \text{Cons}(\Sigma \setminus R, \mu)$.  

**Example 4.** According to the Example 1, $\Psi$ is syntactically represented by $(\Sigma, \preceq_{\Sigma})$ and revising by $\mu$ using the weak comparator gives:

$\text{Bel}(\Psi \circ_{\Sigma} \mu) = \text{Cons}(\{b, \neg a \lor b, \neg a \lor b\}) \cup \text{Cons}(\{a \lor \neg b, \neg a \lor b, \neg a \lor b\})$ and $\preceq_{\Sigma, \mu}$ as shown by Fig. 3.

$$R_3 \equiv_{w} R_4 \equiv_{w} R_5 \equiv_{w} R_6$$

$$\downarrow \downarrow \downarrow$$

$$R_1 \uparrow \uparrow$$

$$R_0 \uparrow \downarrow$$

Fig. 2. Partial preorder on the potential removed sets.

$$a \quad a \lor \neg b$$

$$\downarrow \quad \downarrow$$

$$b \quad \neg a \lor b$$

$$\neg a \lor b$$

Fig. 3. $\preceq_{\Sigma, \mu}$. 

5
PPRSR semantic approach

According to a semantic point of view, the epistemic state $\Psi$ can be equivalently represented by a partial preorder on interpretations such that $\text{Mod}(\text{Bel}(\Psi))$ is minimal in this preorder. We now construct a partial preorder on interpretations applying the comparator to the set of formulae of $\Sigma$ falsified by the interpretations. Let $\omega$ be an interpretation, $F_{\Sigma}(\omega)$ denotes the set of formulae of $\Sigma$ falsified by $\omega$. More formally,

**Definition 12.** Let $\Psi$ be an epistemic state syntactically represented by $(\Sigma, \preceq_\Sigma)$, the partial preorder $\preceq_\Psi$ on interpretations is such that:

$$\forall \omega, \omega' \in W, \quad \omega \preceq_\Psi \omega' \iff F_{\Sigma}(\omega) \subseteq C F_{\Sigma}(\omega').$$

Using this definition, the semantic representation of $\Psi$ is $(W, \preceq_\Psi)$ and is such that $\text{Mod}(\text{Bel}(\Psi)) = \min(W, \preceq_\Psi)$. We are now able to define the semantic counterpart of PPRSR as follows:

**Definition 13.** Let $(W, \preceq_\Psi)$ be a semantic representation of $\Psi$, the revision of $\Psi$ by a formula $\mu$ leads to the semantically revised epistemic state $\Psi_\preceq_\Psi \mu$ such that

$$\text{Mod}(\text{Bel}(\Psi_\preceq_\Psi \mu)) = \min(\text{Mod}(\mu), \preceq_\Psi).$$

The revised epistemic state is semantically represented by $(W, \preceq_\Psi \mu)$, where $\preceq_\Psi \mu$ is a new partial preorder on interpretations, as illustrated in Fig. 4a). This partial preorder is defined from the sets of formulas belonging to $\Sigma \cup \{\mu\}$ falsified by the interpretations. More formally:

**Definition 14.** Let $(\Sigma \preceq C \mu, \preceq_{\Sigma \preceq C \mu})$ be the representation of the syntactically revised epistemic state. The partial preorder on interpretations $\preceq_{\Psi_\preceq_\Psi \mu}$ is such that

$$\forall \omega, \omega' \in W, \quad \omega \preceq_{\Psi_\preceq_\Psi \mu} \omega' \iff F_{\Sigma \preceq C \mu}(\omega) \subseteq C F_{\Sigma \preceq C \mu}(\omega').$$

PPRSR agrees with the notion of P-faithful assignment and the following proposition holds.

**Proposition 1.** Let $\Psi$ be an epistemic state and $\preceq_\Psi$ be a partial preorder on $W$ associated to $\Psi$. Then, $\preceq_\Psi$ is a P-faithful assignment.

The proof of the Proposition 1 (provided in Appendix A) follows from the construction of partial preorder $\preceq_\Psi$.

According to the Proposition 1, the revision operation $\Psi_\preceq_\Psi \mu$ satisfies the postulates $P_1 - P_7$ proposed in [6] that extend the KM-postulates to the revision of partially preordered belief bases.

3.2. PPRSR in the case of the weak comparator

We now focus on the weak comparator $\preceq_\w$ defined in Section 2.3. We show that the semantic counterpart captures the extension of possibilistic revision to partial preorders. Moreover, we refine the semantic counterpart given in Definition 13, in order to provide an equivalence between the PPRSR syntactic and semantic approaches.

When we select the weak comparator the PPRSR framework can capture the possibilistic revision, illustrated in Fig. 4b, recalled in Section 2.3 and the following proposition holds.

**Proposition 2.** Let $\sigma_\pi$ be the possibilistic revision operator.

$$\forall \omega, \omega' \in W, \quad \omega \preceq_\w \omega' \iff \omega \preceq_\Psi \omega' \mu.$$

![Fig. 4. The syntactic and semantic representation of the revised epistemic state.](image-url)
Definition 12 with the weak comparator, we construct a partial preorder on interpretations. The sets of formulae of the semantic representation of \( W \) are illustrated in Table 1 and the partial preorder is such that \( \text{Mod}(\text{Bel}(\Psi)) = \min(\mathcal{W}, \preceq^{\Psi}_{\text{w}}) = \{ \omega_3 \} \).

<table>
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<tr>
<th>( \omega_i )</th>
<th>( a )</th>
<th>( b )</th>
<th>( F_{\Sigma(\omega_i)} )</th>
<th>( F_{\Sigma_{\text{w}}, \mu}(\omega_i) )</th>
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</tbody>
</table>

Example 5. Let \((\Sigma, \preceq_{\Sigma})\) be the syntactic representation of \( \Psi \) from the Example 1 with \( \Sigma = \{a, b, a \lor -b, -a \lor b\} \). Using the Definition 12 with the weak comparator, we construct a partial preorder on interpretations. The sets of formulae of \( \Sigma \) falsified by the interpretations is illustrated in Table 1 and the partial preorder \( \preceq^{\Sigma}_{\text{w}} \) is given by the Fig. 5a. Therefore \( (\mathcal{W}, \preceq^{\Psi}_{\text{w}}) \) is the semantic representation of \( \Psi \) and is such that \( \text{Mod}(\text{Bel}(\Psi)) = \min(\mathcal{W}, \preceq^{\Psi}_{\text{w}}) = \{ \omega_3 \} \).

Let \((\Sigma_{\text{w}}, \mu_{\text{w}}, \preceq_{\Sigma_{\text{w}}, \mu_{\text{w}}})\) be the syntactic representation of the epistemic state \( \Psi \) revised by \( \mu \). Using the Definition 12 with the weak comparator, we construct a new partial preorder on the interpretations. The sets of formulae of \( \Sigma_{\text{w}}, \mu_{\text{w}}, \preceq_{\Sigma_{\text{w}}, \mu_{\text{w}}} \) falsified by the interpretations are illustrated in Table 1 and the partial preorder \( \preceq^{\Psi}_{\text{w}, \text{sem}} \) is given by the Fig. 5b. Therefore \( (\mathcal{W}, \preceq^{\Psi}_{\text{w}, \text{sem}}) \) is the semantic representation of \( \Psi \) revised by \( \mu \) and with the Proposition 3 is such that \( \text{Mod}(\text{Bel}(\Psi_{\text{w}, \mu})) = \min(\text{Mod}(\mu), \preceq^{\Psi}_{\text{w}}) = \{ \omega_0, \omega_1 \} \).

If we apply, the semantic possibilistic revision of \((\mathcal{W}, \preceq^{\Psi}_{\text{w}})\) by \( \mu \) which preserves the relative ordering between the models of \( \mu \) and considers all the counter-models of \( \mu \) as impossible, we obtain the partial preorder \( \preceq^{\Psi}_{\text{w}, \mu} \) illustrated in Fig. 5c. Therefore \( (\mathcal{W}, \preceq^{\Psi}_{\text{w}, \mu}) = (\mathcal{W}, \preceq^{\Psi}_{\text{w}, \mu}) \).

As illustrated in the following example, the semantic counterpart defined in the general case does not provide the equivalence between syntactic and semantic approaches.

Example 6. We consider the following counter-example. Let \( \Phi \) be an epistemic state and \((\Gamma, \preceq_{\Gamma})\) be the syntactic representation of \( \Phi \) such that \( \Gamma = \{a, b, -c\} \) and \( \preceq_{\Gamma} \) is illustrated by the Fig. 6a. We revise \( \Gamma \) by \( \mu = a \to b \land c \). We obtain the following syntactic representation of the revised epistemic state: \( \Gamma_{\text{w}, \mu} = \{a, b, -c, a \to b \land c\} \) and \( \preceq^{\Gamma}_{\text{w}, \mu} \) is illustrated by the Fig. 6b.

From a syntactic point of view, we obtain two removed sets \( R_0 = \{a\} \) and \( R_1 = \{-c\} \) and the belief set corresponding to the revised epistemic state is \( \text{Bel}(\Phi_{\text{w}, \mu}) = \text{Cons}(\{b, -c, a \to b \land c\}) \lor \text{Cons}(\{a, b, a \to b \land c\}) \). From a semantic point of view, the
sets of formulae falsified by the interpretations are illustrated in Table 2. We construct the partial preorder $\succeq_{\Phi}$ according to the Definition 12 and the partial preorder $\succeq_{\Phi_{\omega_{\text{sem}}}}$ according to Definition 14 as it is illustrated in Fig. 7.

From the partial preorder $\succeq_{\Phi_{\omega_{\text{sem}}}}$ and from the Definition 13, we have $\text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}})) = \{\omega_0, \omega_2, \omega_7\}$. But the models of $\text{Bel}(\Phi_{\omega_{\text{sem}}})$ are $\omega_2$ and $\omega_7$. So, the syntactic and the semantic approaches are not equivalent.

In order to obtain the equivalence between syntactic and semantic approaches, we refine the semantic counterpart. When defining the removed sets within the syntactic approach, we consider minimal (with respect to inclusion) subsets of formulae, this minimality has to be taken into account in the semantic approach too.

**Definition 15.** Let $(\mathcal{W}, \succeq_{\Phi})$ be a semantic representation of $\Psi$, the revision of $\Psi$ by a formula $\mu$ leads to the semantically revised epistemic state $\Psi_{\omega_{\text{sem}}}$ such that

$$\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}})) = \{\omega | \omega \in \min(\text{Mod}(\mu), \succeq_{\Phi}) \}\text{ and } \exists \omega', \text{ if } F_{\omega_{\text{sem}}}(\omega') \subset F_{\omega_{\text{sem}}}(\omega)$$.

The equivalence between the semantic and the syntactic PPRSR is given by the following proposition.

**Proposition 3.** Let $\Psi_{\omega_{\text{sem}}}$ be the syntactically revised epistemic state by a formula $\mu$ and let $\Psi_{\omega_{\text{sem}}}$ be the semantically revised epistemic state by a formula $\mu$.

$$\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}})) = \text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}}))$$.

Sketch of proof of Proposition 3 (provided in Appendix A): follows from the definition of $\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}}))$.

**Example 7.** We first consider the Exemple 1, in this context, we have $\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}})) = \{\omega_0, \omega_1\}$ and $\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}})) = \{\omega_0, \omega_1\}$.

The Definition 15 has no effect on $\text{Mod}(\text{Bel}(\Psi_{\omega_{\text{sem}}}))$ because the sets of formulae of $\Sigma \cup \{\mu\}$ falsified by $\omega_0$ and $\omega_1$ are minimal with respect to inclusion.

We now consider the Exemple 6, we have $\text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}})) = \{\omega_2, \omega_7\}$ and $\text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}})) = \{\omega_0, \omega_2, \omega_7\}$. If we apply the Definition 15, $\text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}})) = \{\omega_2, \omega_7\}$ because $F_{\omega_{\text{sem}}}(\omega_2) \subset F_{\omega_{\text{sem}}}(\omega_0)$. We have $\text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}})) = \text{Mod}(\text{Bel}(\Phi_{\omega_{\text{sem}}}))$.

### 4. Encoding PPRSR in answer set programming

As mentioned in the previous section, the original RSR approach was proposed for revising belief bases consisting of formulas in CNF. RSR was formalized in terms of answer set programming by the construction of a logic program with the same
spirit of [39] and the smodels system [44] was adapted in order to compute the preferred answer sets corresponding to the
removed sets [8]. More recently, the extension of RSR to fusion, Removed Sets Fusion (RSF) has been proposed for merging
belief bases consisting of arbitrary propositional formulae, and a suitable encoding in answer programming enables the
computation of removed sets with any ASP solver equipped with the minimize statement [27]. We now extend this method
to the revision of partially preordered information.

We first translate our revision problem into a logic program with answer sets semantics, denoted by \( \Pi_{S, \mu} \). The set of
answer sets is denoted by \( S(\Pi_{S, \mu}) \). We then define a partial preorder between answer sets of \( \Pi_{S, \mu} \) and we show a
one-to-one correspondence between removed sets of \( S \cup \{ \mu \} \) and preferred answer sets of \( \Pi_{S, \mu} \).

Let \( \Sigma \) be a set of partially preordered formulae and \( \mu \) a formula such that \( \Sigma \cup \{ \mu \} \) is inconsistent. The set of all positive
literals of \( \Pi_{S, \mu} \) is denoted by \( V^+ \) and the set of all negative literals of \( \Pi_{S, \mu} \) is denoted by \( V^- \). The set of all rule atoms
representing formulae is defined by \( R^+ = \{ r_f \in \Sigma \} \) and \( F(\{ r_f \}) \) represents the formula of \( \Sigma \) corresponding to \( r_f \) in \( \Pi_{S, \mu} \), namely \( \forall v \in R^+, F(\{ r_f \}) = f \). This translation requires the introduction of intermediary atoms representing subformulae of \( f \). We denote
by \( \rho_f \) the intermediary atom representing \( f \) which is a subformula of \( f \in \Sigma \). To each answer set \( S \) of \( \Pi_{S, \mu} \), an interpretation
of \( \Sigma \cup \{ \mu \} \) is associated. Each interpretation of \( \Sigma \cup \{ \mu \} \) corresponds to several potential removed sets denoted by \( F_0(R^+ \cap S) \).

1. In the first step, we introduce rules in order to build a one-to-one correspondence between answer sets of \( \Pi_{S, \mu} \) and
interpretations of \( V^- \). For each atom, \( a \in V^- \) two rules are introduced: \( a \leftarrow \neg a' \) and \( a' \leftarrow \neg a \) where \( a' \in V^- \) is the nega-
tive atom corresponding to \( a \).

2. In the second step, we introduce rules in order to exclude the answer sets \( S \) corresponding to interpretations which are
not models of \( (\Sigma \setminus F) \cup \{ \mu \} \) with \( F = \{ r_f \} \in S \). According to the syntax of \( f \), the following rules are introduced:
- If \( f = \neg a \), the rule \( r_f \leftarrow \neg a \) is introduced;
- If \( f = \neg \neg f \), the rule \( r_f \leftarrow \neg \rho_f \) is introduced;
- If \( f = \neg \neg f \), the rule \( r_f \leftarrow \rho_f \), \ldots, \( r_f \leftarrow \rho_f \) is introduced;
- If \( f = \neg \neg f \) \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& \& …
Definition 16. Let $\preceq_{\Sigma}$ be a partial preorder on $\Sigma, \mu$ be a formula such that $\Sigma \cup \{\mu\}$ is inconsistent, $S \in S(\Pi_{\Sigma,\{\mu\}})$. $S$ is a preferred answer set of $\Pi_{\Sigma,\{\mu\}}$ if:

- $\exists S' \in S(\Pi_{\Sigma,\{\mu\}})$ such that $F_\diamond(S' \cap R') \subset F_\diamond(S \cap R')$.
- $\exists S' \in S(\Pi_{\Sigma,\{\mu\}})$ such that $F_\diamond(S' \cap R') \supset F_\diamond(S \cap R')$.

Example 10. We have $F_\diamond(S_0 \cap R') \supset F_\diamond(S_1 \cap R')$ and $F_\diamond(S_2 \cap R') \supset F_\diamond(S_1 \cap R')$. So, $S_0(\Pi_{\Sigma,\{\mu\}}) = \{S_0, S_2\}$.

Remark. As previously, it is possible to refine the notion of preferred answer set with an extra preference according to a strategy $P$. Let $S_x, S_y \in S_P(\Pi_{\Sigma,\{\mu\}})$, $S_y$ is preferred to $S_x$ according to $\text{CARD}$ (resp. $\text{MIN}$) iff $|F_\diamond(S_y \cap R')| \leq |F_\diamond(S_x \cap R')|$ (resp. $|F_\diamond(S_y \cap R') \cap \text{MIN}| \leq |F_\diamond(S_x \cap R') \cap \text{MIN}|$).

Example 11. We have $S_0$ is as preferred as $S_2$ according to $\text{CARD}$ and $S_0$ is preferred to $S_2$ according to $\text{MIN}$.

The one-to-one correspondence between preferred answer sets of $\Pi_{\Sigma,\{\mu\}}$ and the removed sets is given by the following proposition:

Proposition 6. Let $\Sigma$ be a finite set of partially preordered formulae and $\mu$ be a formula such that $\Sigma \cup \{\mu\}$ is inconsistent. $X$ is a removed set of $\Sigma \cup \{\mu\}$ if and only if there exists a preferred answer set $S$ of $\Pi_{\Sigma,\{\mu\}}$ such that $F_\diamond(R' \cap S) = X$.

Sketch of the proof of the Proposition 6 (provided in Appendix A): we show that the set of removed sets of $\Sigma \cup \{\mu\}$ equals the set of preferred answer sets of $\Pi_{\Sigma,\{\mu\}}$.

Example 12. We have $F_\diamond(S_0 \cap R') = \{a, a \lor \neg b\}$ and $F_\diamond(S_2 \cap R') = \{a, b\}$ which correspond to the removed sets $R_0$ and $R_1$ found in the previous section.

Performing PPRSR The theoretical computational complexity of the decision problem “is the formula phi a weak consequence of $\Sigma \cup \{\mu\}$?” is not already known, however a lower bound $\Omega(n^2)$ and an upper bound $O(n^9)$ have been provided in [7]. However, some applications, as illustrated in the following section, require the revision of partially preordered belief bases. We choose the removed set approach rather than the dual one based on preferred maximal consistent subbases because the removed set approach allows for an efficient implementation with ASP. Within the framework of the VENUS project, we conducted an experimental study on an inconsistency handling method stemming from removed sets for partially preordered archaeological information and we obtained interesting results. The practical complexity was reasonable, and therefore encouraged us to investigate further in this direction, as well as to test on revision problem.

Regarding the implementation, CLASP [23] gives us the answer sets of $\Pi_{\Sigma,\{\mu\}}$. But our method requires to partially preorder the answer sets with the comparator $\preceq\omegaP$ to obtain the preferred answer sets corresponding to removed sets. This step is not yet implemented in ASP. We used a java program to partially preorder the answer sets to obtain the preferred answer sets. We denote by $N$ the number of answer sets given by CLASP. The computation of the partial preorder between them can be realized in less than $\frac{n(n-1)}{2}$ comparisons. Indeed, as it showed in [32], it is sufficient to compare the minimal formulae according to $\preceq_{\Sigma}$ of each answer set and so using the following proposition, we reduce the cost of the computation.

Proposition 7. Let $\preceq_{\Sigma}$ be a partial preorder on $\Sigma$, $\mu$ be a formula such that $\Sigma \cup \{\mu\}$ is inconsistent and $S, S' \in S(\Pi_{\Sigma,\{\mu\}})$. $F_\diamond(S \cap R') \preceq_{\omegaP} F_\diamond(S' \cap R')$ iff $\forall y \in \text{Min}(F_\diamond(S \cap R'), \preceq_{\Sigma}) \exists x \in \text{Min}(F_\diamond(S' \cap R'), \preceq_{\Sigma})$ such that $x \preceq_{\Sigma} y$ where $\text{Min}(F_\diamond(S \cap R'), \preceq_{\Sigma}) = \{x | x \in F_\diamond(S \cap R'), \exists y \in F_\diamond(S \cap R'), y <_{\Sigma} x\}$.

Moreover, the determination of the minimal answer sets according to this partial preorder does not increase the computational cost since CLASP and SAT both belong to the NP-complete complexity class.

5. VENUS application

The European VENUS project (Virtual ExploratioN of Underwater Sites) no (IST-034924)3 aims at providing scientific methodologies and technological tools for the virtual exploration of deep underwater archaeology sites. In this context, technologies like photogrammetry are used for data acquisition and the knowledge about the studied objects is provided by both archaeology and photogrammetry. We constructed an application ontology in [43] from a domain ontology which describes the vocabulary on the amphorae (the studied artefacts) and from a task ontology describing the data acquisition process (Fig. 8). This ontology consists of a set of concepts, relations, attributes and constraints like domain constraints: an amphora must have only one typology and for example, this typology is either short Dressel 2-4 or long Dressel 2-4. Our knowledge base contains our ontology and

3 http://www.venus-project.eu.
observations. The ontology represents the generic knowledge which is preferred to observations. The observations on the same amphora can be preordered according to the reliability of the experts who provide them. In this context, we revise the observations by more reliable observations. We only consider a small part of the ontology (Fig. 9) and some observations in order to provide a very simple example where the knowledge base is expressed in propositional logic.

We use the following propositional variables: \( m_i \) for the measurable item, \( a_r \) for the archaeological item, \( a_i \) for the amphora item, \( a \) for the amphora, \( m_1, m_2 \) for the metrologies, \( d_s \) for the short Dressel 2-4 typology, \( d_l \) for the long Dressel 2-4 typology, \( h_{m_1}, h_{m_2} \) for has_metrology, \( h_1, h_2 \) for the total heights, \( l_1, l_2 \) for the total lengths. The propositional translation of the extract of the ontology can be resumed by the set of formulae:

\[
G = \{ a \rightarrow a_i \land (d_s \lor d_l), a_i \rightarrow a_r, a_r \rightarrow m_i, m_i \rightarrow h_{m_i} \lor h_{m_2}, h_{m_i} \rightarrow m_1, h_{m_2} \rightarrow m_2, m_1 \rightarrow l_1 \land h_1, m_2 \rightarrow l_2 \land h_2. \\
(d_s \lor d_l) \land (\neg d_s \lor \neg d_l) \}
\]

Then we have two sets of observations provided by two different experts. The observations of the first expert \( \{a, d_s, l_1, h_1\} \) lead to the instance denoted by

\[
I_1 = \{a, a_i, a_r, d_s, m_1, m_i, h_{m_1}, l_1, h_1\}
\]

and the observations of the second one is \( \{a, d_l, l_2, h_2\} \) lead to the instance denoted by:

\[
I_2 = \{a, a_i, a_r, d_l, m_2, m_i, h_{m_2}, l_2, h_2\}
\]

By hypothesis, the ontology and the constraints which are also called the generic knowledge cannot be modified. Moreover, we consider that the second expert is more reliable than the first one. We revised the first observations \( \Sigma = I_1 \setminus (I_1 \cap I_2) \) by
\[ M = G \cup I_2 \text{ where } G \text{ is the generic knowledge and } I_2 \text{ is the second set of observations and the revised preorder is represented by Fig. 10 and we obtain } \Sigma_{\alpha \prec M} M = \text{Cons}(\Sigma \setminus R \cup M) \text{ with } R = \{d_i\}. \]

The revision presented in the Section 3 is the first step of the revision to apply in the VENUS context. Indeed, the revision could be defined as follows:

- \[ \Sigma_{\alpha \prec M} M = \bigwedge_{R \subseteq R_1(\Sigma,M)} \text{Cons}(\Sigma \setminus R \cup M) \text{ with } \Sigma = I_1 \setminus (I_1 \cap I_2) \text{ and } M = G \cup I_2. \]
- \[ \Sigma_{\alpha \prec M} M : \]
  - (i) \[ \forall \psi, \phi \in M, \psi \prec_{\Sigma_{\alpha \prec M}} \phi \iff \psi \preceq_M \phi. \]
  - (ii) \[ \forall \psi, \phi \in \Sigma, \psi \prec_{\Sigma_{\alpha \prec M}} \phi \iff \psi \preceq_{\Sigma} \phi. \]
  - (iii) \[ \forall \psi \in \Sigma, \mu \in M, \mu \prec_{\Sigma_{\alpha \prec M}} \psi. \]

6. Conclusion

This paper presents a new framework for revising partially preordered information called Partially Preordered Removed Sets Revision (PPRSR) which extends the Removed Sets approach to partial preorders. The paper shows that PPRSR can be successfully encoded into answer set programming and proposes an implementation stemming from ASP solvers. It shows that the extension of the possibilistic revision to partial preorders can be captured within the PPRSR framework allowing for an efficient implementation with ASP. It illustrates how PPRSR can be applied within the context of the VENUS project dealing with archaeological information. An experimental study has now to be conducted in the context of the VENUS project in order to provide a more accurate evaluation of the performance of PPRSR. We have to deeper investigate the use of ASP solver statements in order to directly define a partial preorder between answer sets. A future work will investigate the use of the lexicographic comparator for defining revision operations within the framework of PPRSR.

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Appendix A. proofs

**Proposition 1.** Let $\Psi$ be an epistemic state and $\preceq_\Psi$ be a partial preorder on $\mathcal{W}$ associated to $\Psi$. Then, $\preceq_\Psi$ is a P-faithful assignment.

**Proof.** Let $\Sigma$ be a set of formulae. The preorder on the interpretations is defined as follows: $\forall \omega, \omega' \in \mathcal{W}$, $\omega \preceq_\Psi \omega'$ iff $F_\Sigma(\omega) \subseteq F_\Sigma(\omega')$. We want to show that $\preceq_\Psi$ is a P-faithful assignment.

1. We show that if $\omega, \omega' \in \text{Mod}(Bel(\Psi))$ then $\omega \preceq_\Psi \omega'$ is impossible.
   Let $\omega, \omega' \in \text{Mod}(Bel(\Psi))$, then $F_\Sigma(\omega) = \emptyset$ and $F_\Sigma(\omega') = \emptyset$. So, we have $F_\Sigma(\omega) = \emptyset$ and $\omega \preceq_\Psi \omega'$.

2. We show that if $\omega \not\in \text{Bel}(\Psi)$ then there exists $\omega \in \mathcal{W}$ such that $\omega \not\in \text{Bel}(\Psi)$ and $\omega \preceq_\Psi \omega'$.
   Let $\omega \not\in \text{Bel}(\Psi)$ then $F_\Sigma(\omega) = \emptyset$. We show that there exists $\omega$ such that $\omega \not\in \text{Bel}(\Psi)$ and $\omega \preceq_\Psi \omega'$.

$\Psi$ is an epistemic state, represented by $(\Sigma, \preceq_\Sigma)$. So there exists $\omega$ such that $\omega \not\in \text{Bel}(\Psi)$ and $F_\Sigma(\omega) = \emptyset$. We have $F_\Sigma(\omega') \neq \emptyset$ and $F_\Sigma(\omega) \subseteq F_\Sigma(\omega')$. Therefore, $\omega \preceq_\Psi \omega'$.

3. We show that if $\Psi = \Phi$ then $\preceq_\Psi = \preceq_\Phi$.
   Let $\Psi$ be an epistemic state and let $\preceq_\Psi$ be the associated partial preorder on the interpretations. If $\Psi = \Phi$ then $\preceq_\Psi = \preceq_\Phi$. \Box

**Proposition 2.** Let $\omega_\Psi$ be the possibilistic revision operator.

$\forall \omega, \omega' \in \mathcal{W}$, $\omega \preceq_\Psi \omega'$ iff $\omega \preceq_\Psi \omega'$.

**Proof (Possibilistic equivalence).** We can distinguish three cases:

1. We want to show that:
   If $\omega \in \text{Mod}(\mu)$ and $\omega' \in \text{Mod}(\mu)$ then $\omega \preceq_\Psi \omega'$ iff $\omega' \preceq_\Psi \omega'$.
   We have:

   $\omega \preceq_\Psi \omega'$ iff $F_\Sigma(\omega) \subseteq F_\Sigma(\omega')$ such that $\phi \preceq_\Sigma \mu \phi$.

   Since $\omega \in \text{Mod}(\mu)$ and $\omega' \in \text{Mod}(\mu)$ therefore $\mu \not\in \text{Mod}(\omega)$ and $\mu \not\in \text{Mod}(\omega')$. According to the Definition 8 of $\text{Mod}(\mu)$, $\psi \in \Sigma$ such that $\psi \preceq_\Sigma \mu \phi$ iff $\psi \preceq_\Sigma \psi$.

   Thus:

   $\omega \preceq_\Psi \omega'$ iff $\forall \phi \in F_\Sigma(\omega), \exists \phi \in F_\Sigma(\omega')$.

   Since $\Sigma_{\text{mod}} = \Sigma \cup \{\mu\}$:

   $\omega \preceq_\Psi \omega'$ iff $\forall \phi \in F_\Sigma(\omega), \exists \phi \in F_\Sigma(\omega')$ such that $\psi \preceq_\Sigma \phi$.

   Moreover, since $\omega \in \text{Mod}(\mu)$ and $\omega' \in \text{Mod}(\mu)$ thus $\mu \not\in \text{Mod}(\omega)$ and $\mu \not\in \text{Mod}(\omega')$. Therefore $F_\Sigma(\omega) = F_\Sigma(\omega)$ and $F_\Sigma(\omega') = F_\Sigma(\omega')$.

   $\omega \preceq_\Psi \omega'$ iff $\forall \phi \in F_\Sigma(\omega), \exists \phi \in F_\Sigma(\omega')$ such that $\psi \preceq_\Sigma \psi$.

   We want to show that:

   If $\omega \not\in \text{Mod}(\mu)$ and $\omega' \not\in \text{Mod}(\mu)$ then $\omega \not\in \text{Mod}(\mu)$.

   We have $\omega \not\in \text{Mod}(\mu)$ and $\omega' \not\in \text{Mod}(\mu)$ then $\omega$ and $\omega'$ falsify the formula $\mu$ so $\mu \in F_\Sigma(\omega)$ and $\mu \in F_\Sigma(\omega')$. According to the definition of the partial preorder $\leq_{\Sigma_{\text{mod}}} \psi$, we have:

   $\forall \phi \in F_\Sigma(\omega), \exists \phi \in F_\Sigma(\omega')$ (in this case $\mu$) such that $\psi \preceq_{\Sigma_{\text{mod}}} \mu \phi$ and thus $\omega \preceq_{\Psi_{\Sigma_{\text{mod}}} \mu} \omega'$. By symmetry, we have $\omega' \preceq_{\Psi_{\Sigma_{\text{mod}}} \mu} \omega$ therefore $\omega \preceq_{\Psi_{\Sigma_{\text{mod}}} \mu} \omega'$.

3. We want to show that:

   If $\omega \in \text{Mod}(\mu)$ and $\omega' \not\in \text{Mod}(\mu)$ then $\omega \preceq_{\Psi_{\Sigma_{\text{mod}}} \mu} \omega'$. 


We have $\omega \in \text{Mod}(\mu)$ and $\omega' \not\in \text{Mod}(\mu)$ then $\omega$ does not falsify the formula $\mu$ and $\omega'$ falsifies the formula $\mu$ thus $\mu \not\in F_{\Sigma_{\omega}^{\omega'}(\omega)}$ and $\mu \in F_{\Sigma_{\omega}^{\omega'}(\omega')}$. We have: $\forall \phi \in F_{\Sigma_{\omega}^{\omega'}(\omega)}, \exists \phi \in F_{\Sigma_{\omega}^{\omega'}(\omega')}$ (in this case $\mu$) such that $\phi \not\in F_{\Sigma_{\omega}^{\omega'}(\omega)}$ and so $\omega^w_{\Sigma_{\omega}^{\omega'}(\omega')}\omega'$. \hfill $\square$

**Proposition 3.** Let $\Psi_{\omega}^{\omega'}(\mu)$ be the syntactically revised epistemic state by a formula $\mu$ and let $\Psi_{\omega}^{\omega'}(\mu)$ be the semantically revised epistemic state by a formula $\mu$.

$$\text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))) = \text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))).$$

**Proof.** We want to show that $\text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))) = \text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu)))$, according to the Definition 15, $\text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))) = \{\omega|\omega \in \min(\text{Mod}(\mu), \leq_{\omega'}) \text{ and } \not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)\}$. We show that

$$\text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))) = \{\omega|\omega \in \min(\text{Mod}(\mu), \leq_{\omega'}) \text{ and } \not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)\}.$$

1. We show that:

$$\text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))) \subseteq \{\omega|\omega \in \min(\text{Mod}(\mu), \leq_{\omega'}) \text{ and } \not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)\}.$$

Let $F_{\Sigma_{\omega}^{\omega'}}(\omega)$ be the set of formulae falsified by $\omega$. Since $\omega \in \text{Mod}(\Sigma \cup \{\mu\})$ thus $\omega$ does not falsify the formulae from $\Sigma \cup \{\mu\}$ but $\omega$ falsifies the formulae belonging to the removed set $R$. Therefore $F_{\Sigma_{\omega}^{\omega'}}(\omega) = R$. We want to show that:

(a) $\omega \in \min(\text{Mod}(\mu), \leq_{\omega'})$. Suppose $\omega \not\in \min(\text{Mod}(\mu), \leq_{\omega'})$ there exists $\omega' \in \text{Mod}(\mu)$ such that $\omega'\not\leq_{\omega'}\omega$. By Definition 12, $\omega' \not\leq_{\omega'}\omega$ iff $F_{\Sigma_{\omega}^{\omega'}}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}}(\omega)$. However $F_{\Sigma_{\omega}^{\omega'}}(\omega) = R$ thus $F_{\Sigma_{\omega}^{\omega'}}(\omega') \subset R$ which contradicts the fact that $R$ is a removed set.

(b) $\not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)$.

2. We show that:

$$\{\omega|\omega \in \min(\text{Mod}(\mu), \leq_{\omega'}) \text{ and } \not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)\} \subseteq \text{Mod}(\text{Bel}(\Psi_{\omega}^{\omega'}(\mu))).$$

We show that:

(a) $\not\exists \omega' \in \text{Mod}(\mu)$ such that $(\Sigma \setminus F_{\Sigma_{\omega}^{\omega'}(\omega)}) \cup \{\mu\}$ is inconsistent and $F_{\Sigma_{\omega}^{\omega'}(\omega)} = R$. Suppose there exists $\omega' \in \text{Mod}(\mu)$ such that $(\Sigma \setminus F_{\Sigma_{\omega}^{\omega'}(\omega)}) \cup \{\mu\}$ is consistent and $F_{\Sigma_{\omega}^{\omega'}(\omega)} \subset F_{\Sigma_{\omega}^{\omega'}(\omega)}$. This contradicts the starting hypothesis: $\omega \in \min(\text{Mod}(\mu), \leq_{\omega'})$ and $\not\exists \omega', F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega') \subset F_{\Sigma_{\omega}^{\omega'}(\mu)}(\omega)$.

(b) $\not\exists \omega' \in \text{Mod}(\mu)$ such that $(\Sigma \setminus F_{\Sigma_{\omega}^{\omega'}(\omega)}) \cup \{\mu\}$ is inconsistent and $F_{\Sigma_{\omega}^{\omega'}(\omega)} \cap \{\mu\}$.

Suppose there exists $\omega' \in \text{Mod}(\mu)$ such that $(\Sigma \setminus F_{\Sigma_{\omega}^{\omega'}(\omega)}) \cup \{\mu\}$ is consistent and $F_{\Sigma_{\omega}^{\omega'}(\omega)} \cap \{\mu\}$. According to Definition 12, $\omega' \not\leq_{\omega'}\omega$ donc $\omega \not\in \min(\text{Mod}(\mu), \leq_{\omega'})$ which contradicts the starting hypothesis. \hfill $\square$

**Proposition 4.** Let $\rho$ a rule atom or an intermediary atom.

$$\rho \in \text{CN}(\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}) \text{ iff } l \not\in F_{\omega}(\rho).$$

**Proof.** The proof is obvious according the rules construction. \hfill $\square$

**Proposition 5.** Let $\Sigma$ be a set of partially preordered formulae. Let $S \subseteq V$ be a set of atoms. $S$ is an answer set of $\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}$ iff $S$ corresponds to an interpretation $I_{S}$ of $V'$ which satisfies $(\Sigma \setminus F_{\omega}(R' \cap S)) \cup \{\mu\}$.

**Proof.**

1. We first show that if $S$ is an answer set of $\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}$ then $I_{S}$ is an interpretation of $V'$ which satisfies $\Sigma \setminus F_{\omega}(R' \cap S) \cup \{\mu\}$. Let $S$ be an answer set of $\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}$. We have $I_{S} = \{a | a \in S \cup \{a \not\in S\}\}$ that is to say if $a \in S$ then $a \in I_{S}$ and if $a' \in S$ then $\neg a \in I_{S}$.

- We first show that $I_{S}$ is an interpretation of $V'$.
  - To do that, we show the mutual exclusion that is to say $\forall a \in V'$, either $a \in S$ or $a' \in S$ because:
    - If $a \notin S$ and $a' \notin S$ then the rule $a \leftarrow not a'$ is applied and $a \in \text{CN}(\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}$, yet $S$ is an answer set of $\Pi_{\Sigma_{\omega}^{\omega'}(\mu)}$ thus $S = \text{CN}(\Pi_{\Sigma_{\omega}^{\omega'}(\mu)})$; hence $a \in S$ which contradicts the hypothesis.
    - If $a \in S$ and $a' \in S$ then the only rule allowing to deduce $a$ is $a \leftarrow not a'$. For that $a$ is true, it is necessary that $a'$ is false, thus $a' \notin S$, which contradicts the hypothesis.
We show too that false \not\in S and contradiction \not\in S. Suppose that false \in S, there are two cases:

* Let contradiction \not\in S, the rule contradiction \not\rightarrow false, not contradiction is applied and contradiction \in CN(Pi_{\Sigma_{\mu}}), yet S is an answer set of \Pi_{\Sigma_{\mu}} thus S = CN(Pi_{\Sigma_{\mu}}) thus contradiction \in S which contradicts the initial hypothesis.

* Let contradiction \in S. The only rule allowing to deduce contradiction is contradiction \not\rightarrow false, not contradiction. In order to have contradiction true, it is necessary that false is true and contradiction is false thus contradiction \not\in S which contradicts the hypothesis.

Thus, I_{k} is an interpretation of V'.

- We then show that I_{k} satisfies \Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\} that is to say I_{k} \in Mod(\Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\}).

  - We show that I_{k} is a model of \mu. Suppose that I_{k} \not\in Mod(\mu) then I_{k} \not\models \mu. The rule introduced by (3), we get false \in CN(Pi_{\Sigma_{\mu}}), yet S is an answer set of \Pi_{\Sigma_{\mu}} thus S = CN(Pi_{\Sigma_{\mu}}) thus false \in S which is impossible since false \not\in S. Thus I_{k} \not\models \mu.

- Suppose that I_{k} \not\in Mod(\Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\}). Then there exists a formula f \in \Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\} such that I_{k} \not\models f. Since I_{k} \not\models f then f \not\models = \mu thus f \in \Sigma \setminus F_{0}(R' \cap S). According to the rule introduced by the step 2 concerning f, we get r_{f} \in CN(Pi_{\Sigma_{\mu}}) yet S is an answer set of \Pi_{\Sigma_{\mu}} thus S = CN(Pi_{\Sigma_{\mu}}) r_{f} \in S. If r_{f} \in S then f \in F_{0}(R' \cap S) and thus f \not\models \Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\} which contradicts the hypothesis.

As a consequence, I_{k} satisfies \Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\}.

2. Then, we show that, if I_{k} is an interpretation of V' which satisfies \Sigma \setminus F_{0}(R' \cap S) \cup \{\mu\} then S is an answer set of \Pi_{\Sigma_{\mu}}. From I_{k}, we build a set S such that S = \{a \mid a \in I_{k}\} \cup \{a' \mid a \not\in I_{k}\} that is to say if a \in I_{k} then a \in S and if a \not\in I_{k} then a' \in S. We then add rule atoms and intermediate atoms. We show that S = CN(Pi_{\Sigma_{\mu}}).

- We first show that S \subseteq CN(Pi_{\Sigma_{\mu}}). Let a \in S be an atom. There are two cases:

  - Let a \in V' I_{k} is an interpretation of V' +, and thus a' \not\in S. The rule a \not\models not a' implies that a \in CN(Pi_{\Sigma_{\mu}}).

  - Let a \in V. The proof is similar.

Moreover, if \rho is a rule atom or an intermediary atom. We know that if \rho \in S then \rho \in CN(Pi_{\Sigma_{\mu}}) according to the Proposition 4.

- Then we show that CN(Pi_{\Sigma_{\mu}}) \subseteq S. Clearly, we have CN(Pi_{\Sigma_{\mu}}) \subseteq V \cup \{false, contradiction\}.

  - We first show that CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') \subseteq S. Suppose that there exists a \in CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') such that a \not\in S. There are two cases:

    * Let a \in V'.a' \in S because I_{k} is an interpretation of V' +, and we know that S \subseteq CN(Pi_{\Sigma_{\mu}}) thus a' \in CN(Pi_{\Sigma_{\mu}}). But the only rule allowing the deduction of a is a \not\models not a' thus a \not\models CN(Pi_{\Sigma_{\mu}})which is impossible since a \in CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R'). Thus a \not\in S.

    * Let a \in V. The proof is similar.

Moreover, if \rho is a rule atom or an intermediary atom such that \rho \in CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') and \rho \not\in S. According to the Proposition 4, we know that \rho \in CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') implies that I_{k} \not\models F_{0}(\rho). But, if \rho \not\in S then I_{k} \not\models F_{0}(\rho) which is contradictory.

- Then, we show that CN(Pi_{\Sigma_{\mu}}) \cap R' \subseteq S. Suppose that there exists a \in CN(Pi_{\Sigma_{\mu}}) \cap R' such that a \not\in S. Then a = r_{f} thus r_{f} \in S and I_{k} is an interpretation of V'. The only rule concerning f is introduced in 2. The body of the rule belongs to CN(Pi_{\Sigma_{\mu}}). Since CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') \subseteq S then it belongs to S too. That implies that I_{k} \not\models f. But I_{k} is a model of \Sigma \setminus F_{0}(R' \cap S) and r_{f} \not\in S which contradicts the hypothesis.

Moreover, if \rho is a rule atom or an intermediary atom such that \rho \in CN(Pi_{\Sigma_{\mu}}) \cap R' and \rho \not\in S. According to the Proposition 4, we know that \rho \in CN(Pi_{\Sigma_{\mu}}) \cap R' implies that I_{k} \not\models F_{0}(\rho). But, if \rho \not\in S then I_{k} \not\models F_{0}(\rho) which is contradictory.

- Finally, we show that false \not\in CN(Pi_{\Sigma_{\mu}}). Suppose that false \in CN(Pi_{\Sigma_{\mu}}), then the rule introduced in 3 is such that its body belongs to CN(Pi_{\Sigma_{\mu}}) and since CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') \subseteq S then it belongs to S too. But that is to say that the formula \mu is not satisfied by I_{k} which is impossible since I_{k} is a model of \mu. Thus false \not\in CN(Pi_{\Sigma_{\mu}}) and by consequence contradiction \not\in CN(Pi_{\Sigma_{\mu}}).

We have CN(Pi_{\Sigma_{\mu}}) \subseteq V \cup \{false, contradiction\} and we have shown that CN(Pi_{\Sigma_{\mu}}) \cap (V \setminus R') \subseteq S, CN(Pi_{\Sigma_{\mu}}) \cap R' \subseteq S, false \not\in CN(Pi_{\Sigma_{\mu}}) and contradiction \not\in CN(Pi_{\Sigma_{\mu}}) thus

CN(Pi_{\Sigma_{\mu}}) \subseteq S. □
Proposition 6. Let \( \Sigma \) be a finite set of partially preordered formulae and \( \mu \) be a formula such that \( \Sigma \cup \{ \mu \} \) is inconsistent. \( X \) is a removed set of \( \Sigma \cup \{ \mu \} \) if there exists a preferred answer set \( S \) of \( \Pi_{\Sigma \cup \{ \mu \}} \) such that \( F_0(R^* \cap S) = X \).

To proof this proposition, we use the following intermediary results which are a consequence of the Proposition 5:

Proposition 8. Let \( R \subseteq \Sigma \). If \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent then there exists a set of atoms \( S \) such that \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \) and \( F_0(R^* \cap S) \subseteq R \).

Proof. Let \( R \subseteq \Sigma \) be such that \( (\Sigma \setminus R) \cup \{ \mu \} \) be consistent. We show that there exists a set of atoms \( S \) such that \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \) and \( F_0(R^* \cap S) \subseteq R \).

We can build a set of atoms \( S \) such that \( S = \{ a \mid a \in R \} \cup \{ a' \mid a' \notin R \} \). From this interpretation \( I_k \), \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \) and \( R \) contains all the formulae falsified by the interpretation \( I_k \). But \( R \) may also contain formulae satisfied by \( I_k \). There is no one-to-one correspondence between answer sets and potential removed sets. An answer set may correspond to several potential removed sets. □

Corollary 1. Let \( R \subseteq \Sigma \) be such that \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent. If \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent, then there exists a set of atoms \( S \) such that \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \) and \( F_0(R^* \cap S) = R \).

Proposition 9. If there exists a set of atoms \( S \) such that \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \) then \( (\Sigma \setminus F_0(R^* \cap S)) \cup \{ \mu \} \) is consistent.

Proof. Let \( S \) be a set of atoms such that \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \). We show that \( (\Sigma \setminus F_0(R^* \cap S)) \cup \{ \mu \} \) is consistent.

From \( S \), we build an interpretation \( I_k = \{ a \mid a \in S \} \cup \{ a' \mid a' \notin S \} \). According to the Proposition 5, we know that all the rules \( r_j \) which do not appear in \( S \) correspond to formulae \( f \) satisfied by the interpretation \( I_k \). Thus \( (\Sigma \setminus F_0(R^* \cap S)) \cup \{ \mu \} \) is consistent. □

Proof (One-to-one correspondence). Let \( S_n(\Pi_{\Sigma \cup \{ \mu \}}) \) be the set of preferred answer sets of \( \Pi_{\Sigma \cup \{ \mu \}} \) according to the comparator \( \leq_c \). Let \( R_n(\Sigma \cup \{ \mu \}) \) be the set of removed sets of \( \Sigma \cup \{ \mu \} \). We want to show that: \( \{ F_0(R' \cap S) \mid S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \} = R_n(\Sigma \cup \{ \mu \}) \).

1. We show that \( \{ F_0(R' \cap S) \mid S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \} \subseteq R_n(\Sigma \cup \{ \mu \}) \). Let \( S \) be a preferred answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \), i.e. \( S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \) such that \( R = F_0(R^* \cap S) \). We want to show that \( R \in R_n(\Sigma \cup \{ \mu \}) \) i.e.:
   - \( R \) is a potential removed set of \( \Sigma \cup \{ \mu \} \) and \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent.
   - \( R \) is minimal according to the inclusion: \( R \subseteq \Sigma \) such that \( (\Sigma \setminus R) \cup \{ \mu \} \) and \( R \subseteq R' \subset \Sigma \).
   - \( R \) is preferred according to the comparator \( \leq_c : R \subseteq \Sigma \) such that \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent and \( R' \subseteq R \).

   According to the Proposition 9, \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent. Suppose that \( R \not\subseteq R_n(\Sigma \cup \{ \mu \}) \) then, we have two possibilities:
   - (a) \( R \) is not minimal according to the inclusion: \( \exists R' \subseteq \Sigma \) such that \( (\Sigma \setminus R') \cup \{ \mu \} \) and \( R' \not\subseteq R \).
   - (b) \( R \) is not preferred according to the comparator \( \leq_c : (\Sigma \setminus R) \cup \{ \mu \} \) and \( \exists R' \subseteq \Sigma \) such that \( (\Sigma \setminus R') \cup \{ \mu \} \) and \( R' \not\subseteq R \).

   Let \( S' = \{ a \mid a \in R \} \cup \{ a' \mid a' \notin R \} \). Clearly, \( I_k = \{ a \mid a \in S \} \cup \{ a' \mid a' \notin S \} \) is an interpretation of \( V' \) which satisfies \( (\Sigma \setminus R') \cup \{ \mu \} \) and \( R' = F_0(R' \cap S') \). According to the Corollary 1, \( S' \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \). Moreover, \( R \not\subseteq R_n(\Sigma \cup \{ \mu \}) \).
   - (a) \( R \subseteq R_n(\Sigma \cup \{ \mu \}) \) and \( F_0(R' \cap S') \subseteq R_n(\Sigma \cup \{ \mu \}) \) which contradicts the hypothesis \( S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \).
   - (b) \( R_n(\Sigma \cup \{ \mu \}) \) then \( F_0(R' \cap S') \subseteq R_n(\Sigma \cup \{ \mu \}) \) which contradicts the hypothesis \( S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \).

   Thus \( R \in R_n(\Sigma \cup \{ \mu \}) \).

2. We show that \( R_n(\Sigma \cup \{ \mu \}) \subseteq \{ F_0(R' \cap S) \mid S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \} \). Let \( R \) be a removed set of \( \Sigma \cup \{ \mu \} \) according to the comparator \( \leq_c \). We have \( (\Sigma \setminus R) \cup \{ \mu \} \) is consistent, so, there exists an interpretation \( I_k \) which satisfies \( (\Sigma \setminus R) \cup \{ \mu \} \) and \( S = \{ a \mid a \in R \} \cup \{ a' \mid a' \notin R \} \) be a set of atoms. Clearly, \( R = F_0(R^* \cap S) \). According to the Corollary 1, \( S \) is an answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \). Suppose that \( S \) is not a preferred answer set of \( \Pi_{\Sigma \cup \{ \mu \}} \), i.e. \( S \not\subseteq S_n(\Pi_{\Sigma \cup \{ \mu \}}) \). Then we have two possibilities:
   - (a) \( S \) is not minimal according to the inclusion: \( \exists S' \in S(\Pi_{\Sigma \cup \{ \mu \}}) \) such that \( F_0(S' \cap R') \subseteq F_0(S \cap R') \). Since \( R = F_0(R^* \cap S) \), \( F_0(S \cap R') \subseteq R \). Moreover, \( F_0(S \cap R') \subseteq \Sigma \) and \( (\Sigma \setminus F_0(S \cap R')) \cup \{ \mu \} \) is consistent. So \( R \) is not a removed set of \( \Sigma \cup \{ \mu \} \) which contradicts the hypothesis.
   - (b) \( S \) is not preferred according to the comparator \( \leq_c : S \subseteq S(\Pi_{\Sigma \cup \{ \mu \}}) \) such that \( F_0(S \cap R') \subseteq F_0(S \cap R') \). As \( R = F_0(R^* - F_0(R^* \cap S))F_0(S \cap R') \). Moreover, \( F_0(S \cap R') \subseteq \Sigma \) and \( (\Sigma \setminus F_0(S \cap R')) \cup \{ \mu \} \) is consistent. So, \( R \) is not a removed set of \( \Sigma \cup \{ \mu \} \) which contradicts the hypothesis.

Therefore, \( S \in S_n(\Pi_{\Sigma \cup \{ \mu \}}) \). □
References