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RESEARCH ARTICLE

Variational Principles with Generalized Distances and the Modelization of Organizational Change

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This paper has a two-fold focus. The mathematical aspect of the paper shows that new and existing quasimetric and weak τ-distance versions of Ekeland's variational principle are equivalent in the sense that one implies the other, and so are their corresponding fixed-point results. The practical aspect of the paper, using a recent variational rationality approach of human behavior, offers a model of organizational change, where generalized distances model inertia in terms of resistance to change. The formation and breaking of routines relative to hiring and firing workers will be used to illustrate the obtained results.

Keywords: Ekeland's variational principle, Caristi's fixed-point theorem, quasimetric, w-distance, τ-distance, variational rationality.

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1. Introduction

Ekeland's variational principle (abbrev. EVP) [1] is one of the most important results in nonlinear analysis; it allows us to study minimization problems in the case where the lower level set of a minimization problem is not compact. Roughly speaking, starting at an approximate solution of a lower semicontinuous function, we can find a better approximate solution of the given function which is the unique solution of a perturbed function defined as a sum of the given function and a 'weighted' distance. Although many extensions of EVP have been given, very few justification on the importance and need of such extensions in practical applications exist except [2] in Computer Sciences and [3, 4] in Behavioral Sciences.

A recent variational rationality approach in human behaviors dynamics [5, 6] has shown how EVP represents a benchmark case to examine how agents and organizations, before succeeding to reach their final goals, must accept to follow a succession of worthwhile stays and changes. A change is worthwhile if motivation to change is proportionally higher than resistance to change; the reverse for a stay.

On the application side, this paper offers a model of organizational change where generalized distances model inertia in terms of resistance to change. It is illustrated by the formation and breaking of routines relative to hiring and firing workers.

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By now, a large number of equivalences and extensions of EVP are known in the literature. On one hand, the original EVP is equivalent to Caristi's fixed-point theorem (abbrev. CFPT) [7], Takahashi's minimal point theorem [8], the nonempty intersection theorem [9], the drop theorem [10], the petal theorem [11], the equilibrium theorem [9], etc. On the other hand, its equivalents and itself have been extensively generalized in many directions in order to fit a new setting or a particular application including to vector- and set-valued cost functions, quasimetrics, w-distances, τ-distances, etc.

In [12], Bao and Khanh proved that many generalizations of EVP including Zhong's result in [13] are equivalent to the original EVP. Since then, many new extensions of EVP with w-distances, τ-distances, τ-functions, and weak τ-functions were added into the literature; the distance used in [13] is of any kind of these generalized distances.

In [14], Kada, Suzuki and Takahashi introduced a generalized distance (called by them *w-distance*) and proved that both EVP and CFPT hold when a w-distance plays the role of the metric in the original result. Several years later, Suzuki [15] generalized both w-distance and Tataru's distance [16] to τ-distance and justified that the Banach fixed-point theorem (known also as the contraction principle) holds with τ-distances. Recently, Lin and Du [17] established a τ-function version of EVP while Khanh and Quy [18, 19] formulated a weak τ-function version. All of these results were studied in complete metric spaces.

In this paper, we revisite the question "Are several recent generalizations of Ekeland's variational principle more general than the original principle?" in [12]. We do even more. We show that almost all (if not ALL) versions of EVP with generalized distances in a underlying metric space can be extended to a quasimetric setting and that they are equivalent to the quasimetric version of EVP in [4, Corollary 3.3]. It is worth mentioning that the topology induced by a quasimetric is not automatic to be Hausdorff and that such a requirement was missed in the formulation and proof of a quasimetric generalization of EVP in [20].

This paper has a two-fold focus: the equivalence between a quasimetric version of EVP and a weak τ-distance one and the need of such extensions for possible applications to the formation and breaking of routines relative to hiring and firing workers. The rest of the paper is organized as follows. Section 2 presents preliminaries on quasimetrics, w-distances, τ-functions, τ-distances, and weak τ-distances. Section 3 contains two more 'general' but equivalent versions of EVP corresponding to quasimetrics and weak τ-distances. One of them is equivalent to the other and to many existing versions of EVP in the literature. Section 4 is devoted to formulate generalized versions of CFPT which are equivalent to the corresponding EVP results obtained in Section 3. Section 5 focuses attention on behavioral motivations for generalized versions of EVP and what the obtained results add to applications in Behavioral Sciences.

2. Preliminaries

In this section, we present several kinds of generalized distances used in variational principles in order to model resistance to change in the context of organizational change. Let us recall the definition of quasimetric spaces and notions of convergence, closedness, limit, completeness, and Hausdorff topological property in these spaces; cf. [3, 4, 20, 21].

Definition 1 **(quasimetrics and metrics).** A bifunction $q : X \times X \to \mathbb{R}_+ := [0, +\infty)$ on a nonempty set X is said to be a QUASIMETRIC iff for all $x, y, z \in X$ it satisfies

(q1) $q(x, y) \geq 0$ (nonnegativity);
(q2) $q(x, y) = 0 \Longleftrightarrow x = y$ (coincidence axiom);
(q3) $q(x, z) \leq q(x, y) + q(y, z)$ (triangle axiom).

If a quasimetric q enjoys the axiom of symmetry,

(q4) $q(x, y) = q(y, x)$ (axiom of symmetry),

then it is called a METRIC.

We prefer to use d for a metric. A pair (X, q) stands for a quasimetric space X with a quasi-metric q while (X, d) symbols for a metric space X with a metric d. A quasimetric on the real numbers can be defined by

$$q(x, y) = x - y \text{ if } x \geq y, \text{ and } q(x, y) = 1 \text{ otherwise.} \tag{2.1}$$

Consider the topology induced by q with the following base of half-open intevals

$$\mathcal{B} := \{[a, b) |\ a, b \in \mathbb{R},\ a < b\}.$$

This topological space is called the *Sorgenfrey* line. It describes the process of filing down a metal stick: it is easy to reduce its size, but it is difficult or impossible to grow it.

Below are some basic concepts in quasimetric spaces which reduce to the known ones in metric spaces.

Definition 2 **(basic concepts).** Let (X, q) be a quasimetric space, and $\{x_n\}$ a sequence in X.

- The sequence $\{x_n\}$ is said to be LEFT-CONVERGENT to a point $x_* \in X$, denoted by $x_n \to x_*$, iff the quasidistances $q(x_n, x_*)$ tend to zero as $n \to \infty$, i.e. $\lim_{n\to\infty} q(x_n, x_*) = 0$.
- The sequence $\{x_n\}$ is said to be LEFT-CAUCHY iff for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$q(x_n, x_m) < \varepsilon \text{ for all } m \geq n \geq N_\varepsilon.$$

- The quasimetric space (X, q) is said to be LEFT-COMPLETE iff each left-Cauchy sequence is left-convergent.
- The quasimetric space (X, q) is said to be LEFT-HAUSDORFF iff every left-convergent sequence has a unique limit; that is, if there are $x_*, y_* \in X$ such that $\lim_{n\to\infty} q(x_n, x_*) = 0$ and $\lim_{n\to\infty} q(x_n, y_*) = 0$, then one has $x_* = y_*$.

Note that one could define notions of right-convegence, right-Cauchy, right-completeness and right-Hausdorff as well as formulate 'right' results. They seems to add no significant information even though left and right notions and results are different.

Note also that a left-convergent sequence in a quasimetric space is not necessarily left-Cauchy and that a quasimetric space might not be left-Hausdorff; see [4, 22], and the references therein. It was proved in [22, Example 3.16] that the bifunction

q on $X = [0,1]$ with

$$q(x,y) = \begin{cases} x - y & \text{if } x \geq y \\ 1 + x - y & \text{if } x < y \text{ but } (x,y) \neq (0,1) \\ 1 & \text{if } (x,y) = (0,1) \end{cases}$$

is a quasimetric without the left-Hausdorff property. Indeed, the sequence $\{x_n\}$ with $x_n = \frac{1}{n}$ has two limits $x_* = 0$ and $y_* = 1$.

Note finally that the quasimetric space $(\mathbb{R}, q)$ with q in (2.1) is left-Cauchy and left-Hausdorff.

Next, we recall definitions of several generalized distances used in the paper.

Definition 3 **(w-distances, [14, Kada et al.]).** Let (X,d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called a w-DISTANCE on X iff it satisfies

(w1) $p(x,z) \leq p(x,y) + p(y,z)$ (triangularity);
(w2) p is lower semicontinuous in its second variable (lower semicontinuity);
(w3) $\forall \varepsilon > 0,\ \exists\, \delta > 0 :\ p(z,x) \leq \delta\ \wedge\ p(z,y) \leq \delta \implies d(x,y) \leq \varepsilon$.

Definition 4 **(τ-distances [15, Suzuki]).** Let (X,d) be a metric space. A function $p : X \times X \to \mathbb{R}_+$ is called a τ-DISTANCE on X iff there is a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for $x,y,z \in X$ and $t \in \mathbb{R}_+$ the following hold:

($\tau 1'$) $p(x,z) \leq p(x,y) + p(y,z)$ (triangle inequality);
($\tau 2'$) if $x_n \xrightarrow{d} x$ and $\lim_{n\to\infty} \sup_{m \geq n} \eta(z_n, p(z_n, x_m)) = 0$ for some sequence $\{z_n\} \subset X$, then $p(w,x) \leq \lim \inf_{n\to\infty} p(w, x_n)$ for all $w \in X$ (weak lower semicontinuity);
($\tau 3'$) if $\lim_{n\to\infty} \eta(x_n, z_n) = 0$ and $\lim_{n\to\infty} \sup_{m\geq n} p(x_n, y_m) = 0$, then $\lim_{n\to\infty} \eta(y_n, z_n) = 0$ (the uniqueness of (η, p)-convergence);
($\tau 4'$) $\lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$ (the uniqueness of η-convergence);
($\tau 5'$) $\eta(x, 0) = 0$, $\eta(x,t) \geq t$ and $\eta(x, .)$ is concave.

Definition 5 **(τ-functions [17, Lin and Du]).** Let (X,d) be a metric space. A function $p : X \times X \to I\!R_+$ is called a τ-FUNCTION iff the following four conditions hold

($\tau 1$) $p(x,z) \leq p(x,y) + p(y,z)$ (triangle inequality);
($\tau 2$) for all $x \in X$, $p(x, \cdot)$ is lower semicontinuous (lower semicontinuity);
($\tau 3$) for all $\{x_n\}$, $\{y_n\}$ with $\lim_{n\to\infty} p(x_n, y_n) = 0$ and $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$, one has $\lim_{n\to\infty} d(x_n, y_n) = 0$ (p-convergence implies d-convergence);
($\tau 4$) $p(x,y) = 0$ and $p(x,z) = 0$ imply that $y = z$ (indistancy implies coincidence).

Definition 6 **(weak τ-functions [18, 19, Khanh and Quy]).** Let (X,d) be a metric space. A function $p : X \times X \to I\!R_+$ is called a WEAK τ-FUNCTION iff it satisfies conditions ($\tau 1$), ($\tau 3$), and ($\tau 4$) in Definition 5.

It is known that every w-distance is a τ-function; see [17] and that both τ-functions and τ-distances, which are incomparable, are weak τ-functions; see [18].

3. EVPs with Generalized Distances

In this section, we present two versions of EVP in terms of a quasimetric and a weak τ-distance, respectively. One one hand, the latter looks more general than the former and is better fitted to applications in Behavioral Sciences. On the other hand, they are equivalent in the sense that one could be derived from the other.

THEOREM 3.1 **(EVP [1]).** *Let (X, d) be a complete metric space, $\varphi : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ a function being lower semicontinuous, bounded below, and not identically equal to $+\infty$, $\varepsilon > 0$, and $x_0 \in \operatorname{dom}\varphi$ an ε-minimal solution of φ, i.e. $\varphi(x_0) \le \inf_{x\in X} \varphi(x) + \varepsilon$. Then, for every $\lambda > 0$, there exists a point $x_* \in \operatorname{dom}\varphi$ such that*

(i) $\varphi(x_*) \le \varphi(x_0)$;
(ii) $d(x_0, x_*) \le \lambda$;
(iii) $\varphi(x) + (\varepsilon/\lambda)d(x_*, x) > \varphi(x_*),\ \forall\, x \ne x_*$.

In [13], Zhong proved that the conclusions of EVP still hold when the metric d is replaced by the following generalized distance $p : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$p(x, y) := \frac{d(x, y)}{1 + h(d(x_0, x))}, \tag{3.2}$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and satisfies $\int_0^\infty dt(1 + h(t)) < +\infty$. The class of generalized distances in (3.2) has been enlarged to various broarder classes of w-distances [14], Tataru's distances [16], τ-functions [17], and weak τ-functions [18, 19]; see also the bibiographies therein.

In [17], Lin and Du established a generalized version of EVP for decreasingly-closed (known also as lower semicontinuous from above) functions and τ-functions in complete metric spaces which was further extended to weak τ-functions by Khanh and Quy [18, 19] .

Recently, Bao et al. formulated several set-valued version of EVP acting from a quasimetric space to a vector space equipped with a variable ordering structure in [3, 4, 21]. Theorem 3.2 below is a simplest version for extended-real-valued functions.

Definition 7 **(decreasing left-lower-semicontinuity).** A function $\varphi : X \to \overline{\mathbb{R}}$ is said to be DECREASINGLY LEFT-LOWER-SEMICONTINUOUS iff for any left-convergent sequence $\{x_n\}$ in $\operatorname{dom}\varphi$ with $\varphi(x_{n+1}) \le \varphi(x_n)$, $\forall n \in \mathbb{N}$, if $\lim_{n\to\infty} q(x_n, x_*) = 0$, then $\varphi(x_*) \le \lim_{n\to\infty} \varphi(x_n)$.

In the metric setting, this property was considered by Kirk and Saliga [23] (called by them *lower semicontinuity from above*) meaning that for every sequence $\{x_n\}$ satisfying $\varphi(x_{n+1}) \le \varphi(x_n)$ and $x_n \to x_*$, one has $\varphi(x_*) \le \lim_{n\to\infty} \varphi(x_n)$ and by Qiu [24] (called by him *sequentially lower monotonicity*) meaning that if a sequence $\{x_n\}$ converges to x_* and satisfies $\varphi(x_{n+1}) \le \varphi(x_n)$ for all $n \in \mathbb{N}$, then $\varphi(x_*) \le \varphi(x_n)$ for all $n \in \mathbb{N}$.

The diference beween lower-semicontinuity and decreasing lower-semicontinuity is obvious; the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = 1$ for $x \le 0$ and $-x$ otherwise in the complete metric space $(\mathbb{R}, |\cdot|)$ is decreasingly lower-semicontinuous, but not lower-semicontinuous at $x = 0$. In general, it is not difficult to check that if φ is lower semicontinuous (known also as level-closed) on $\operatorname{dom}\varphi \setminus \operatorname{Max}(\varphi)$, then it is decreasingly left-lower-semicontinuous on $\operatorname{dom}\varphi$, where $\operatorname{Max}(\varphi)$ is the collection of all the local maxima of φ.

THEOREM 3.2 **(a quasimetric version of EVP [4, Corollary 3.3]).** *Let (X, q) be a left-complete and left-Hausdorff quasimetric space, and $\varphi : X \to \overline{\mathbb{R}}$ be a decreasingly left-lower-semicontinuous and bounded from below function on X with $\operatorname{dom}\varphi \neq \emptyset$. For any $\lambda > 0$ and $x_0 \in \operatorname{dom}\varphi$, there is $x_* \in \operatorname{dom}\varphi$ such that*

(i) $\varphi(x_*) + \lambda q(x_0, x_*) \leq \varphi(x_0)$;
(ii) $\varphi(x) + \lambda q(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_*$.

Associate with φ, q, and λ a set-valued mapping $S_{\varphi,q,\lambda} : X \rightrightarrows X$ defined by

$$S_{\varphi,q,\lambda}(x) := \{y \in X |\ \varphi(y) + \lambda q(x, y) \leq \varphi(x)\}, \tag{3.3}$$

conclusions (i) and (ii) of Theorem 3.2 could be read as there is $x_* \in S_{\varphi,q,\lambda}(x_0)$ such that $S_{\varphi,q,\lambda}(x_*) = \{x_*\}$.

Let us provide an example which satisfies all assumptions of Theorem 3.2, but does not fulfill many hypotheses of Theorem 3.1. Consider the quasimetric $(\mathbb{R}, q)$ where q is defined by (2.1) and construct a function $\varphi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\varphi(x) := \begin{cases} 1 & \text{if } \ x \leq 0 \\ x & \text{if } \ x > 0. \end{cases}$$

Next, we will show that φ is decreasingly lower-semicontinuous. Fix an arbitrary left-Cauchy sequence $\{x_n\}$. Assume that it is left-convergent to x_*, i.e., $\lim\limits_{n\to\infty} q(x_n, x_*) = 0$ and the sequence $\{\varphi(x_n)\}$ is decreasing, i.e., $\varphi(x_{n+1}) \leq \varphi(x_n)$ for all $n \in \mathbb{N}$. Without loss of generality we may assume that $q(x_n, x_*) < 1$ and $q(x_n, x_{n+1}) < 1$ for all $n \in \mathbb{N}$. Then, we have

$$x_n \geq x_*,\ q(x_n, x_*) = x_n - x_*,\ x_n \geq x_{n+1}, \ \text{ and } \ \varphi(x_{n+1}) \leq \varphi(x_n)$$

yielding that $x_n \leq 0$ and $\varphi(x_n) = 0$ for all $n \in \mathbb{N}$; $x^* \leq 0$ and $\varphi(x_*) = 0$; and $\varphi(x_*) = \lim\limits_{n\to\infty} \varphi(x_k)$. The latter equality ensures that φ is decreasingly lower-semicontinuous. It is not dificult to check that the quasimetric version of EVP holds for φ and (X, q) under consideration. When $\lambda = 1$, we have $S_{\varphi,q,1}(x^*) = \{x^*\}$ for all $x^* \in \mathbb{R}$.

Technically, the decreasingly left-lower-semicontinuity assumption of the cost function φ in Theorem 3.2 could be weakened to the so-called limiting monotonicity property of the set-valued mapping $S_{\varphi,q,\lambda}$; known also as *dynamical closedness* in [25]. The reader could find examples in [18, 19] which illutrate that such an effort is worth doing.

THEOREM 3.3 **(an enhanced quasimetric version of EVP).** *Let (X, q) be a left-complete and left-Hausdorff quasimetric space and $\varphi : X \to \overline{\mathbb{R}}$ a bounded from below function with $\operatorname{dom}\varphi \neq \emptyset$, which is not necessarily decreasingly left-lower-semicontinuous. Given $\lambda > 0$. Assume that the mapping $S_{\varphi,q,\lambda}$ defined by (3.3) enjoys the limiting monotonicity condition:*

> *For every left-Cauchy sequence $\{x_n\}$ such that $x_{n+1} \in S_{\varphi,q,\lambda}(x_n), \forall n \in \mathbb{N}$ and $\{x_n\}$ is left-convergent to a limit x_*, one has $x_* \in S_{\varphi,q,\lambda}(x_n)$, $\forall\, n \in \mathbb{N}$.*

Then, for every $x_0 \in \operatorname{dom}\varphi$, there exists $x_* \in \operatorname{dom}\varphi$ such that conclusions (i) *and*

(ii) *in Theorem 3.2 hold.*

Proof. Starting from $x_0 \in \operatorname{dom}\varphi$, we recursively construct a sequence $\{x_n\}$ as follow:

$$x_{n+1} \in S_{\varphi,q,\lambda}(x_n) \quad \text{and} \quad q(x_n, x_{n+1}) \geq \sup_{y \in S_{\varphi,q,\lambda}(x_n)} q(x_n, y) - 2^{-n}. \tag{3.4}$$

Then, the following hold:

— $x_n \in S_{\varphi,q,\lambda}(x_n)$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
— $S_{\varphi,q,\lambda}(y) \subset S_{\varphi,q,\lambda}(x)$ for all $y \in S_{\varphi,q,\lambda}(x)$;
— $S_{\varphi,q,\lambda}(x_m) \subset S_{\varphi,q,\lambda}(x_n)$ for all $m, n \in \mathbb{N}_0$ with $m > n$.

Summing up the inequalities in (3.3) with $x = x_i$ and $y = x_{i+1}$ for $i = 0, \ldots, n$, one has

$$\sum_{i=0}^{n} q(x_i, x_{i+1}) \leq \frac{1}{\lambda}\big(\varphi(x_0) - \varphi(x_n)\big) \leq \frac{1}{\lambda}\big(\varphi(x_0) - \inf_{x \in X} \varphi(x)\big) < +\infty,$$

where the last estimate holds due to the boundedness from below of φ. Passing to limit as $n \to \infty$ one obtains

$$\sum_{i=0}^{\infty} q(x_i, x_{i+1}) < +\infty \quad \text{and} \quad \lim_{n \to \infty} q(x_n, x_{n+1}) = 0$$

clearly verifying that the sequence $\{x_n\}$ is left-Cauchy in the quasimetric space (X, q). Since the space is assumed to be left-complete, the chosen sequence converges to some limit x_*. The limiting monotonicity condition implies $x_* \in S_{\varphi,q,\lambda}(x_n)$ for all $n \in \mathbb{N}_0$, and thus

$$S_{\varphi,q,\lambda}(x_*) \subset S_{\varphi,q,\lambda}(x_n), \ \forall\, n \in \mathbb{N}_0. \tag{3.5}$$

Obviously, (i) holds with $n = 0$. To prove (ii) it is sufficient to show that if $y_* \in S_{\varphi,q,\lambda}(x_*)$, then $y_* = x_*$. Assume that $y_* \in S_{\varphi,q,\lambda}(x_*)$. We get from $y_* \in S_{\varphi,q,\lambda}(x_*)$ that $y_* \in S_{\varphi,q,\lambda}(x_n)$ for all $n \in \mathbb{N}$. Using inequalities in (3.4) we could get an upper estimate for $q(x_n, y_*)$:

$$q(x_n, y_*) \leq \sup_{y \in S_{\varphi,q,\lambda}(x_n)} q(x_n, y) \leq q(x_n, x_{n+1}) + 2^{-n}.$$

Passing to the limit as $n \to \infty$, one has $q(x_n, y_*) \to 0$. This means that y_* is another limit of the left-convergent sequence $\{x_n\}$ in addition to x_*. Since (X, q) is left-Hausdorff, $y_* = x_*$. The proof is complete. □

Obviously, Theorem 3.3 $\implies$ Theorem 3.2 since if φ is decreasingly left-lower-semicontinuous, then $S_{\varphi,q,\lambda}$ has the limiting monotonicity condition; for a proof the reader is referred to, e.g., [18, 21].

Remark 1 **(on directions of generalization).** Since condition (i) with $\lambda = \varepsilon/\lambda$ implies both $\varphi(x_*) \leq \varphi(x_0)$ and $q(x_0, x_*) \leq \lambda$ provided that x_0 is ε-minimal to φ, Theorem 3.2 can be viewed as an extension of Theorem 3.1 in two directions: (a) quasimetric spaces and (b) decreasingly left-lower-semicontinuity.

(a) The extension from the class of metric spaces to that of quasimetric spaces is nontrivial. Since a quasimetric does not enjoy the axiom of symmetry, the diameter of the set $S_{\varphi,q,\lambda}(x_n)$ is not double of its radius and the new recursion in (3.4) is needed to construct a sequence converging to a desired point. Such an extension in quasimetric spaces allows us to apply EVP to applications in Behavioral Sciences in which the cost to change from one state to another is not the same the cost to change back (see the last section for a concrete application).

(b) The modified continuity assumption imposed on the cost function, decreasing left-lower-semicontinuity, is quite technical. With it, each set $S_{\varphi,q,\lambda}(x_n)$ might not be closed in X, but there exists an intersection point of infinitely many sets. See, e.g. [3, 4, 21] and also [18, 19].

Next, we work with generalized distances in quasimetric spaces instead of metric ones.

Definition 8 **(weak τ-distances and τ-functions in a quasimetric space).** Let (X, q) be a quasimetric space. A bifunction $p : X \times X \to \mathbb{R}_+$ is called a WEAK τ-DISTANCE iff it satisfies conditions (τ1), (τ2), and (τ3) in Definition 5 with a quasimetric q instead of a metric d. It is called a WEAK τ-FUNCTION iff it satisfies conditions (τ1) and (τ3) only.

We do not follow the approach used in [17–19]. Ours is based on the following observation.

PROPOSITION 3.4 *Let (X, q) be a left-complete and left-Hausdorff quasimetric space and $p : X \times X \to \mathbb{R}_+$ a weak τ-distance satisfying conditions (τ1), (τ2), and (τ3) in Definition 5 with $d = q$. Define a function $\overline{q} : X \times X \to I\!R_+$ by*

$$\overline{q}(x, y) := \begin{cases} p(x, y) & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y. \end{cases} \tag{3.6}$$

Then: $\overline{q}$ is a quasimetric on X and $(X, \overline{q})$ is a left-complete and left-Hausdorff quasimetric space.

Proof. It is easy to check that $\overline{q}$ is a quasimetric on X. We now justify the left-completeness of the quasimetric space $(X, \overline{q})$. Fix an arbitrary left-Cauchy sequence $\{x_n\}$ in $(X, \overline{q})$, i.e., $\lim_{n\to\infty} \sup_{m>n} \overline{q}(x_n, x_m) = 0$ and consider two cases:

Case 1: $\{x_n\}$ is eventually constant, i.e. there is some integer $M > 0$ such that $x_{M+k} \equiv x_M$ for all $k \in \mathbb{N}$. Obviously, x_M is a limit of the sequence $\{x_n\}$.

Case 2: $\{x_n\}$ is not eventually constant. We may assume that $x_m \neq x_n$ for all $m \neq n$; otherwise, we use a subsequence instead. By the definition of $\overline{q}$ we have $\overline{q}(x_n, x_m) = p(x_n, x_m)$ for all $m > n$. Thus,

$$\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = \lim_{n\to\infty} \sup_{m>n} \overline{q}(x_n, x_m) = 0. \tag{3.7}$$

Fix an arbitrary $k \in \mathbb{N}$. Condition (τ3) for the sequence $\{y_n\}$ with $y_n := x_{n+k}$ for all $n \in \mathbb{N}$ ensures that $\lim_{n\to\infty} q(x_n, x_{n+k}) = 0$, i.e., $\{x_n\}$ is left-Cauchy in the left-complete quasimetric space (X, q). Thus, it is left-convergent to x^* with respect to

q. Using condition (τ2) and the definition of $\overline{q}$, we have

$$\overline{q}(x_n, \overline{x}) \le p(x_n, \overline{x}) \le \liminf_{m\to\infty} p(x_n, x_{n+m}) \ \text{ for all } \ n \in \mathbb{N}$$

clearly verifying that $\overline{q}(x_n, \overline{x})$ tends to zero as $n \to \infty$ due to (3.7). This proves the left-completeness of the space $(X, \overline{q})$.

Next, we show that the left-Hausdorff topological property of the quasimetric space (X, q) implies that of $(X, \overline{q})$. Again, we consider two cases:

Case 1: A constant sequence $\{x_n\}$ with $x_n \equiv y_*$ for all $n \in \mathbb{N}$ is left-convergent to some x_* with respect to $\overline{q}$. If $x_* \ne y_*$, we have $\overline{q}(x_n, x_*) = \overline{q}(y_*, x_*) = p(y_*, x_*) = 0$. ($\tau$3) with $y_n \equiv x_*$ for all $n \in \mathbb{N}$ yields $q(y_*, x_*) = 0$ and thus $y_* = x_*$.

Case 2: A sequence $\{x_n\}$ with $x_m \ne x_n$ for all $m \ne n$ is left-convergent to two limits y_* and x_*. Without loss of generality, we may assume that $x_n \ne y_*$ and $x_n \ne x_*$ for all $n \in \mathbb{N}$. By the definition of $\overline{q}$, we have $\overline{q}(x_n, y_*) = p(x_n, y_*)$ and $\overline{q}(x_n, x_*) = p(x_n, x_*)$ for all $n \in \mathbb{N}$. By (τ3) with $\{x_n\}$ and $\{y_n \equiv y_*\}$, we have $\lim_{n\to\infty} q(x_n, y_*) = 0$. By ($\tau$3) with $\{x_n\}$ and $\{y_n \equiv x_*\}$, we have $\lim_{n\to\infty} q(x_n, x_*) = 0$. Since (X, q) is left-Hausdorff, we have $y_* = x_*$. The proof is complete. □

THEOREM 3.5 **(a weak τ-distance version of EVP).** *Let (X, q) and φ be as in Theorem 3.2 and $p : X \times X \to \mathbb{R}_+$ a weak τ-distance as in Definition 5. For any $\lambda > 0$ and $x_0 \in \operatorname{dom}\varphi$ such that $S_{\varphi,p,\lambda}(x_0) \ne \emptyset$, there exists $x_* \in \operatorname{dom}\varphi$ satisfying*

(i′) $x_* \in S_{\varphi,p,\lambda}(x_0)$, *i.e.* $\varphi(x_*) + \lambda p(x_0, x_*) \le \varphi(x_0)$;
(ii′) $S_{\varphi,p,\lambda}(x_*) \subset \{x_*\}$.

Indeed, Theorem 3.2 $\Longleftrightarrow$ Theorem 3.5 in the sense that one implies the other.

Proof. Since every quasimetric is a weak τ-distance, Theorem 3.5 $\Longrightarrow$ Theorem 3.2 is straightforward. It remains to prove Theorem 3.2 $\Longrightarrow$ Theorem 3.5.

By Proposition 3.4, the quasimetric space $(X, \overline{q})$, where $\overline{q}$ is defined by (3.6), is a left-complete and left-Hausdorff quasimetric space. Employing Theorem 3.2 to the underlying quasimetric space $(X, \overline{q})$, for any $\lambda > 0$ and for each $x_0 \in \operatorname{dom}\varphi$, there exists $x_* \in \operatorname{dom}\varphi$ satisfying

(i) $\varphi(x_*) + \lambda\overline{q}(x_0, x_*) \le \varphi(x_0)$;
(ii) $\varphi(x) + \lambda\overline{q}(x_*, x) > \varphi(x_*)$, $\forall x \ne x_*$.

Obviously, (ii′) holds since $\overline{q}(x_*, x) = p(x_*, x)$, $\forall x \ne x_*$. To verify (i′), we consider two cases:

Case 1: If $x_* \ne x_0$, then $\overline{q}(x_0, x_*) = p(x_0, x_*)$ by (3.6) and thus (i) reduces to (i′).

Case 2: If $x_* = x_0$, then (ii′) ensures that $S_{\varphi,p,\lambda}(x_0) \subset \{x_0\}$. This together with the imposed assumption $S_{\varphi,p,\lambda}(x_0) \ne \emptyset$ implies $S_{\varphi,p,\lambda}(x_0) = \{x_0\}$, i.e. $\varphi(x_*) + \lambda p(x_0, x_*) = \varphi(x_0) + \lambda p(x_0, x_0) \le \varphi(x_0)$ clearly verifying (i′). We, furthermore, get $p(x_0, x_0) = 0$ in this case. □

As a consequence of this result, Theorem 3.1 $\Longleftrightarrow$ [17, Theorem 2.1] $\Longleftrightarrow$ the scalar version of [18, Theorem 3.2].

Next, we provide a weak τ-function version of EVP for functions which might not be decreasingly left-lower-semicontinuous.

THEOREM 3.6 **(an enhanced weak τ-function version of EVP).** *Let (X, q) and φ be as in Theorem 3.3 and $p : X \times X \to \mathbb{R}_+$ a weak τ-function in the* *sense of Definition 5. Assume that the set-valued mapping $S_{\varphi,p,\lambda}$ defined in (3.3) with $q = p$ has the limiting monotonicity condition. Then, for every $\lambda > 0$ and $x_0 \in \operatorname{dom}\varphi$ such that $S_{\varphi,p,\lambda}(x_0) \neq \emptyset$, there exists $x_* \in \operatorname{dom}\varphi$ satisfying (i′) and (ii′) in Theorem 3.5. Indeed, Theorem 3.6 $\Longleftrightarrow$ Theorem 3.3 in the sense that one implies the other.*

Proof. First, we construct a sequence $\{x_n\}$ as follow: for $n \in \mathbb{N}_0$, if $S_{\varphi,p,\lambda}(x_n) = \emptyset$, then set $x_* = x_n$ and STOP; otherwise x_{n+1} is chosen in $S_{\varphi,p,\lambda}(x_n)$ such that

$$p(x_n, x_{n+1}) \geq \sup_{y \in S_{\varphi,p,\lambda}(x_n)} p(x_n, y) - 2^{-n}.$$

Obviously, if $S_{\varphi,p,\lambda}(x_*) = \emptyset$, then x_* satisfies (ii′). Since $x_n \in S_{\varphi,p,\lambda}(x_{n-1}) \subset \cdots \subset S_{\varphi,p,\lambda}(x_0)$, (i′) holds as well.

When $S_{\varphi,p,\lambda}(x_n) \neq \emptyset$ for all $n \in \mathbb{N}_0$, we proceed in the same lines of the proof of Theorem 3.3. □

We conclude this section with a quasimetric version of [17, Theorem 2.1] in quasimetric spaces. Our proof is very simple.

THEOREM 3.7 **(a Λ-distance version of EVP [17, Theorem 2.1].)** *Let (X, q), φ, and p as in Theorem 3.5 and $\Lambda : \mathbb{R} \to \mathbb{R}_{++} := (0, +\infty)$ a* nondecreasing *function. Then, for any $x_0 \in \operatorname{dom}\varphi$ there exists $x_* \in \operatorname{dom}\varphi$ such that*

(i‴) $p(x_0, x_*) \leq \Lambda(\varphi(x_0))\big(\varphi(x_0) - \varphi(x_*)\big)$;

(ii‴) $p(x_*, x) > \Lambda(\varphi(x_*))\big(\varphi(x_*) - \varphi(x)\big),\ \forall\, x \neq x_*$.

Indeed, Theorem 3.7 $\Longleftrightarrow$ Theorem 3.5.

Proof. Observe that if $S_{\varphi,p,\lambda_0} = \emptyset$ with $\lambda_0 = 1/\Lambda(\varphi(x_0)) > 0$, then $x_* = x_0$ satisfies (i‴) and (ii‴). Assume now that it is nonempty and get from Theorem 3.5 for φ, x_0 and λ_0 the following:

(i′) $\varphi(x_*) + \lambda_0 \overline{q}(x_0, x_*) \leq \varphi(x_0)$;
(ii′) $\varphi(x) + \lambda_0 p(x_*, x) = \varphi(x) + \lambda \overline{q}(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_*$.

Since $\overline{q}(x_0, x_*) \geq 0$, (i′) implies $\varphi(x_*) \leq \varphi(x_0)$. Then, the nondecreasing monotonicity of the function Λ gives $\Lambda(\varphi(x_*)) \leq \Lambda(\varphi(x_0))$. This together with (ii′) justifies (ii‴). Details are below:

$$\begin{aligned}
\text{(ii}'\text{)} \quad &\Longleftrightarrow \quad \varphi(x) + \lambda_0 p(x_*, x) > \varphi(x_*), \quad \forall\, x \neq x_* \\
&\overset{\lambda_0 > 0}{\Longleftrightarrow} \quad p(x_*, x) > \frac{1}{\lambda_0}\Big(\varphi(x_*) - \varphi(x)\Big) = \Lambda(\varphi(x_0))\Big(\varphi(x_*) - \varphi(x)\Big), \quad \forall\, x \neq x_* \\
&\overset{monotonicity}{\Longrightarrow} p(x_*, x) > \Lambda(\varphi(x_*))\Big(\varphi(x_*) - \varphi(x)\Big), \quad \forall\, x \neq x_* \quad \Longleftrightarrow \quad \text{(ii}'''\text{)}.
\end{aligned}$$

We have proved that Theorem 3.5 $\Longrightarrow$ Theorem 3.7. By taking $\Lambda(t) \equiv \lambda$ for all $t \in \mathbb{R}$, we also get the validity of the reverse implication. Therefore, the equivalence holds true. □

Remark 2 **(comparisons with known results).** When $q = d$ is a metric, Theorem 3.7 recaptures the result in [17, Theorem 2.1] while Theorem 3.6 improves the corresponding results in [18, 19, 29] for extended real-valued functions. It is important to emphasize that the obtained results are more general than Khanh and Quy's since they worked with weak τ-functions defined in complete metric spaces.

4. Caristi's Fixed-Point Theorems with Generalized Distances

In this section, we formulate quasimetrics and weak τ-distances versions of CFPT. They are equivalent to the corresponding EVP results obtained in Section 3; each version of EVP leads to a quick proof of a related version of CFPT.

First, let us present some developments on CFPT with w-distances and τ-distances.

THEOREM 4.1 **(CFPT [7]).** *Let (X, d) be a complete metric space, $\varphi : X \to \mathbb{R}_+$ a lower semicontinuous function, and $T : X \to X$ a single-valued function satisfying*

$$d(x, T(x)) \leq \varphi(x) - \varphi(Tx),\ \forall x \in X.$$

Then, T has a fixed point.

In [14], Kada et al. introduced w-distances and proved a fixed-point theorem in which a w-distance plays the role of the metric in the original result. They also derived from it many equivalent forms including Subrahmanyam's fixed-point theorem, Kannan's fixed-point theorem, and C̀iriċ's fixed-point theorem.

THEOREM 4.2 **(a w-distance version of CFPT [14, Theorem 4]).** *Let (X, d) be a complete metric space, $p : X \times X \to \mathbb{R}_+$ a w-distance in Definition 3, and $T : X \to X$ a function. Suppose that there exists $r \in [0, 1)$ such that*

$$p(Tx, T^2x) \leq rp(x, Tx),\ \forall\, x \in X \text{ and}$$

$$\inf_{x\in X} \big(p(x, y) + p(x, Tx)\big) > 0,\ \forall\, y \in X \text{ with } \ y \neq Ty.$$

Then, there exists $x_* \in X$ such that $x_* = Tx_*$ and $p(x_*, x_*) = 0$.

In [37], Latif generalized Theorem 4.2 to set-valued mappings $T : X \rightrightarrows X$ and parameter mappings $\Lambda : \mathbb{R}_+ \to \mathbb{R}_{++}$, where $\mathbb{R}_{++} := (0, +\infty)$.

THEOREM 4.3 **(Latif's fixed-point theorem [37, Theorem 2.3]).** *Let X, p, and φ be as in Theorem 4.2 and $T : X \rightrightarrows X$ a set-valued mapping satisfying*

$$\forall\, x \in X,\ \exists\, y \in T(x)\text{: } p(x, y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big),$$

where $\Lambda : \mathbb{R} \to \mathbb{R}_{++}$ is a nondecreasing function. Then, T has a fixed point $x_* \in X$ satisfying $p(x_*, x_*) = 0$.

The above results were further generalized to τ-functions and weak τ-functions settings in, e.g. [17, Theorem 2.2] and [18, Theorem 3.3]. In this paper, we establish two versions of CFPT which are equivalent to the quasimetric and weak τ-distance versions of EVP. Therefore, they are equivalent as well.

THEOREM 4.4 **(a quasimetric version of CFPT).** *Let (X, q) be a left-complete and left-Hausdorff quasimetric space, $\varphi : X \to \overline{\mathbb{R}}$ a function being proper, decreasingly left-lower-semicontinuous, and bounded from below, and $T : X \rightrightarrows X$*

a set-valued mapping. Assume that the set $\Xi := S_{\varphi,q,1}(x_0)$ *is nonempty for some* $x_0 \in \operatorname{dom}\varphi$ *and the pair* (T, Ξ) *satisfies*

$$\forall\, x \in \Xi,\ \exists\, y \in T(x) :\ \varphi(y) + q(x, y) \le \varphi(x). \tag{4.8}$$

Then, T *has a fixed point* $x_* \in \Xi$*, i.e.* $x_* \in T(x_*)$ *satisfying* $p(x_*, x_*) = 0$*. Indeed, Theorem 4.4* $\Longleftrightarrow$ *Theorem 3.2 in the sense that one implies the other.*

Proof. Theorem 3.2 $\Longrightarrow$ Theorem 4.4. By the quasimetric version of EVP in Theorem 3.2, we can find $x_* \in S_{\varphi,q,1}(x_0) = \Xi$ such that $S_{\varphi,q,1}(x_*) \subset \{x_*\}$, where $S_{\varphi,q,1}$ is defined by (3.3). We claim that this point is a fixed point of T. Arguing by contradiction, suppose that it is not true, i.e. $x_* \notin T(x_*)$. This together with condition (4.8) ensures the existence of $y \in T(x_*)$ such that $y \ne x_*$ and $\varphi(y) + q(x_*, y) \le \varphi(x_*)$, i.e. $x_* \ne y \in S_{\varphi,q,1}(x_*) \subset \{x_*\}$. This impossibility verifies that x_* is a fixed point of T. We also get from condition (4.8) that $\varphi(x_*) + q(x_*, x_*) \le \varphi(x_*)$ and thus $q(x_*, x_*) = 0$ due to the nonnegativity property.

Theorem 4.4 $\Longrightarrow$ Theorem 3.2. Assume that all the assumptions in Theorem 3.2 are fulfilled. Given $x_0 \in \operatorname{dom}\varphi$, we consider two cases:

Case 1: there is $x_* \in S_{\varphi,q,\lambda}(x_0)$ such that $S_{\varphi,q,\lambda}(x_*) = \{x_*\}$. Obviously, such an x_* satisfies conditions (i) and (ii) in Theorem 3.2.

Case 2: for every $x \in S_{\varphi,q,\lambda}(x_0)$, there is $y \in S_{\varphi,q,\lambda}(x) \setminus \{x\}$. Observe that if q is a quasimetric in X, so is $q_\lambda := \lambda q$ and that $S_{\varphi,\lambda q,1}(x_0) = S_{\varphi,q,\lambda}(x_0)$. Set $\Xi := S_{\varphi,q_\lambda,1}(x_0)$ and construct a set-valued mapping $T : X \rightrightarrows X$ with

$$T(x) := \{y \in X |\ y \ne x \text{ and } \varphi(y) + q_\lambda(x, y) \le \varphi(x)\} = S_{\varphi,q,\lambda}(x) \setminus \{x\}.$$

Obviously, $\Xi \subset \operatorname{dom} T$ and the pair (Ξ, T) satisfies condition (4.8) in the quasimetric space (X, q_λ). By Theorem 4.4, there is $x_* \in T(x_*)$ contradicting the structure of T. □

Corollary 4.5 **(a weak τ-distance version of CFPT).** *Let* (X, q) *be a left-complete and left-Hausdorff quasimetric space,* $p : X \times X \to \mathbb{R}_+$ *be a weak* τ*-distance in Definition 5, and* $(\tau 3)$*,* $\varphi : X \to I\!R \cup \{+\infty\}$ *be proper, decreasingly left-lower-semicontinuous, and bounded from below. Assume that the set* $\Xi := S_{\varphi,q,1}(x_0)$ *is nonempty for some* $x_0 \in \operatorname{dom}\varphi$ *and the pair* (T, Ξ) *satisfies*

$$\forall\, x \in \Xi,\ \exists\, y \in T(x) :\ \varphi(y) + p(x, y) \le \varphi(x).$$

Then, T *has a fixed point* $x_* \in X$ *such that* $p(x_*, x_*) = 0$.

Indeed, Corollary 4.5 $\Longleftrightarrow$ *Theorem 3.5 in the sense that one implies the other.*

Proof. Proceed similarly the proof of Theorem 4.4; we replace the quasimetric q and the quasimetric version of EVP in Theorem 3.2 by the weak τ-distance p and the weak τ-distance version in Theorem 3.5. □

Corollary 4.6 **(a Λ-distance version of [37, Theorem 2.3]).** *Let* (X, q) *be a left-complete and left-Hausdorff quasimetric space,* $p : X \times X \to \mathbb{R}_+$ *a weak* τ*-distance in Definition 5,* $\varphi : X \to \overline{\mathbb{R}}$ *a function being proper, decreasingly left-lower-semicontinuous, and bounded from below, and* $T : X \rightrightarrows X$ *a set-valued* *mapping. Assume that there is* $x_0 \in \operatorname{dom}\varphi$ *such that the set*

$$\Xi := \{y \in X |\ p(x,y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big)\}$$

is nonempty and that the pair (T, Ξ) *satisfies*

$$\forall\, x \in X,\ \exists\, y \in T(x)\colon p(x,y) \leq \Lambda\big(\varphi(x)\big)\big(\varphi(x) - \varphi(y)\big),$$

where $\Lambda : \mathbb{R} \to \mathbb{R}_{++}$ *is a nondecreasing function. Then, T has a fixed point* $x_* \in X$ *satisfying* $p(x_*, x_*) = 0$.

Indeed, Corollary 4.6 $\Longleftrightarrow$ *Theorem 3.7 in the sense that one implies the other.*

Proof. By using similar arguments of proving the equivalence between a version of EVP and its corresponding version of CFPT. □

We now have that Corollary 4.6 $\Longleftrightarrow$ Theorem 3.7 $\Longleftrightarrow$ Theorem 3.5 $\Longleftrightarrow$ Corollary 4.5. This implies that a weak τ distance version of [37, Theorem 2.3] is not more general than the original form without the monotonicity mapping Λ.

5. Applications to organizational change: hiring and firing routines

In [5, 6], Soubeyran proposed a Variational Rationality (VR) approach which models and unifies a long list of stay/stability and change dynamical systems in Behavioral Sciences in different contexts in many disciplines; e.g. Psychology, Economics, Management Sciences, Decision Theory, Philosophy, Game Theory, Political Sciences, Artificial Intelligence, etc. He has shown how the original EVP, Theorem 3.1, can be seen as a prototype which formalizes, in a crude but nice way, such dynamics as succession of worthwhile stays and changes which balance, each step, motivation- and resistance-to-change to finally end in some variational trap. Recently, Bao el al. considered how set-valued versions of EVP can be applied to the functioning of goal systems in psychology [3] and the capability theory of wellbeing [4]. In this section, we show how quasimetric and w-distance versions of EVP expand the range of applications in Behavioral Sciences.

The literature on formation and breaking routines is enormous and represents a very important area of research. Our behavioral application considers a well known, specific and concrete example of hiring, firing and keeping employment routines within an organization in [26].

A simple model. Consider a hierarchical firm where, each period, an entrepreneur (leader) can hire, keep again and fire numbers of employed workers in l different kinds of skilled and specialized works $x = (x^1, x^2, \ldots, x^l) \in X = \mathbb{R}^l_+$ to produce a quantity $Q(x)$ of a final good of a quality $s(x)$, where $x^j \geq 0$ is the number of employed workers of type $j \in J := \{1, 2, \ldots, l\}$ and the endogenous quality $s(x)$ of this final good depends on the profile of skilled workers x.

The revenue of the entrepreneur is $\varphi\,[Q(x), s(x)]$. His operational cost $\rho(x)$ is the sum of costs to buy the nondurable means used by each worker, and the wages paid to each worker. Then, if, in the last and current periods, the entrepeneur utilizes the profiles x and y of skilled workers, his last and current profits are $g(x) = \varphi\,[Q(x), s(x)] - \rho(x)$ and $g(y) = \varphi\,[Q(y), s(y)] - \rho(y)$.

Let $\overline{g} = \sup\{g(y), y \in X\} < +\infty$ be the maximum profit which the entrepreneur can expect. Then, given the choice of his last and current profiles of workers x and

y, $f(x) = \overline{g} - g(x) \geq 0$ and $f(y) = \overline{g} - g(y) \geq 0$ represent his last and current unsatisfactions to reach his potential maximum profit $\overline{g} < +\infty$.

To be more precise, let us recall the Cobb Douglas production function in the O-Ring theory of the firm—the "O-Ring" terminology comes from the NASA Apollo program, where the quality of the rocket is zero if the quality of only one of its components is zero; see [27]. It is defined by

$$\varphi\left[Q(x), s(x)\right] = k^{\alpha}\left[\prod_{j=1}^{l} s^j x^j\right] L(x)B,$$

where $k > 0$ represents capital (machines), $\alpha > 0$ shows how a marginal increase in the use in capital will increase, more or less, the quality of the rocket, refering to the degree of concavity of the quality of the rocket with respect to the use of capital, $s^j \geq 0$ defines the skill (quality) of each worker of type j (hence the quality of the component he produces), $L(x)$ is the numner of employed workers, and B is the output per worker. The endogenous quality of the final good is $s(x) = \Pi_{j=1}^{l} s^j x^j$. Obviously, $s(x) = 0$ if $s_j = 0$ for, at least, one $j \in J$.

Next, let us present main ingredients and important concepts in the VR approach; see [5, 6] for a general framework.

— n and $n+1$ stands for a last period and a current one.

— $x_n = x$ and $x_{n+1} = y$ refer to a last and a current profile of employed workers.

— $x_n \curvearrowright x_{n+1}$ stands for a current move of the entrepreneur, i.e. he hires, keeps again, and fires $x_{n+1}^j - x_n^j > 0$, $x_{n+1}^j - x_n^j = 0$, and $x_n^j - x_{n+1}^j > 0$ workers of type j, respectively.

— A move is called a **change** iff $x_{n+1}^j \neq x_n^j$ for some $j \in J$.

— A move is called a **stay** iff $x_{n+1}^j = x_n^j$ for all $j \in J$.

— The **advantage-to-change function** $A : X \times X \to \mathbb{R}$ is defined by $A(x, y) = g(y) - g(x)$ as the difference between the profit to use the profile of employed workers y and the profit to use x.

— The advantage-to-change $A(x_n, x_{n+1})$ from an old profile of employed workers x_n to a current profile x_{n+1} is the increase between the current profit $g(x_{n+1})$ to use the new profile of workers x_{n+1} and the profit $g(x_n)$ to use again the last profile of workers x_n.

—If we define $\overline{g} = \sup\{g(x) |\ x \in X\} < +\infty$ as the largest profit the entrepreneur can expect, then the function $f(x) := \overline{g} - g(x)$ measures the residual unsatisfaction to use the profile of employed workers x. Thus, the advantage-to-change from x_n to x_{n+1}

$$A(x_n, x_{n+1}) = f(x_n) - f(x_{n+1}) = [\overline{g} - g(x_n)] - [\overline{g} - g(x_{n+1})] = g(x_{n+1}) - g(x_n)$$

also refer to the difference between the residual unsatisfaction to use again the current profile of workers x_n and the residual unsatisfaction to use the new profile of workers x_{n+1}.

—The **inconvenience-to-change function** $I : X \times X \to \mathbb{R}$ is defined by $I(x, y) = C(x, y) - C(x, x) \geq 0$ as the difference between the cost-to-be-able-to-change $C(x, y) \geq 0$ and the cost-to-be-able-to-stay $C(x, x) \geq 0$. We will show later that, in this example, $C(x, y)$ is the sum of all the costs to be able to hire, keep again and fire workers.

— A move $x \curvearrowright y$ is **worthwhile** to the entrepreneur iff the advantage-to-change from x to y is proportionally bigger than or equal to the inconvenience-to-change up to a prior chosen degree of acceptability $\lambda > 0$, i.e.,

$$g(y) - g(x) \geq \lambda I(x, y) \text{ or } f(x) - f(y) \geq \lambda I(x, y).$$

—The multifunction $W : X \rightrightarrows X$ defined by

$$W_\lambda(x) := \{y \in X |\ A(x, y) \geq \lambda I(x, y)\}$$

is called a **worthwhile** mapping since a move $x \curvearrowright y$ is worthwhile iff $y \in W_\lambda(x)$.
—A **worthwhile transition** $\{x_n\}$ is defined as a succession of worthwhile temporary stays and changes, i.e. $x_{n+1} \in W_\lambda(x_n)$, $\forall\, n \in \mathbb{N}$.
—$x_* \in X$ is called an **aspiration point (strong or weak)** of a worthwhile transition $\{x_n\}$ iff $x_* \in W_\lambda(x_n)$, $\forall\, n \in \mathbb{N}$ (the strong case) or $x_* \in W_\lambda(x_0)$ (the weak case).
—$x_* \in X$ is called a **worthwhile-to-stay trap** iff

$$W_\lambda(x_*) = \{x_*\} \iff g(x) - g(x_*) = f(x_*) - f(x) < \lambda I(x_*, x),\ \forall\, x \neq x_*.$$

This means that being there, it is not worthwhile to move away.
—A **variational trap** is both worthwhile to reach and worthwhile to stay.

The main idea of a behavioral theory is to explain "why, where, how and when" agents perform actions and change, at each current period, along a path of stays and changes $\{x_0, x_1, \ldots, x_n, x_{n+1}, \ldots\}$: i) why the agent, first, has an incentive to take some steps away from his current position and, then, an incentive to stop changing one more step within this period, ii) when, starting from an initial position, a worthwhile transition converges to a variational trap, i.e. it approaches and ends in this trap.

Next, we will provide motivations for us to study extensions of EVP with both quasimetrics and w-distances. In the VR approach in [5, 6], costs-to-be-able-to-change verify, in the simplest prototype case, the following four assumptions:

(1) no change no cost $C(x, x) = 0$, $\forall\, x \in X$;
(2) it is not free to perform a move $C(x, y) > 0$, $\forall\, x \neq y$;
(3) a direct change costs less than any indirect change $C(x, y) \leq C(x, z) + C(z, y)$, $\forall\, x, y, z \in X$;
(4) the cost to perform a change cannot be equal to the cost to undo that change $C(x, y) \neq C(y, x)$, $\forall\, x, y \in X$.

Mathematically, such a cost function is indeed a quasimetric in X.

In the model of the formation and breaking of routines relative to hiring and firing workers, the inconvenience-to-change function can be defined in terms of hiring, keeping again, and firing costs. More presicely, to be able to hire one skilled worker of type j, ready to work, costs $c^j_H(t) \geq 0$. The variable t represents the last t worker of type j to be hired. Notice that this formulation considers that, in the current period, to hire one more worker can be more or less costly, depending on how many workers the entrepreneur have hired before this last one in this current period. These costs include search and training costs. To fire one worker of type j, costs $c^j_F(t) \geq 0$, where t represents the last t worker to be fired. These costs represent separation and compensation costs. To keep a worker, ready to work, one period more, costs $c^j_K(t) \geq 0$. These conservation costs include knowledge regeneration and motivation costs. Then, in the current period, costs to hire $y^j - x^j \geq 0$ (resp., costs to fire and costs to keep) workers of type j are defined by

$$C_H^j(x^j, y^j) = \int_{x^j}^{y^j} c_H^j(t)dt,\ C_F^j(x^j, y^j) = \int_{y^j}^{x^j} c_F^j(t)dt,\ C_K^j(x^j, x^j) = \int_0^{x^j} c_K^j(t)dt,$$

respectively. Then, costs to be able to change from using x^j workers of type j to y^j workers of type j are

$$C^j(x^j, y^j) := \begin{cases} C_K^j(x^j, x^j) + C_H^j(x^j, y^j) & \text{if } y^j \geq x^j, \\ C_K^j(y^j, y^j) + C_F^j(x^j, y^j) & \text{if } y^j \leq x^j, \end{cases}$$

for $j \in J = \{1, 2, \ldots, l\}$.

Observe from the context that costs-to-be-able-to-stay $C^j(x^j, x^j) = C_K^j(x^j, x^j)$ **are strictly positive** since the cost to be able to keep x^j workers of type j does exist. Observe also that C^j does not enjoy the symmetricity property since, in general, we have the inequality

$$C^j(x^j, y^j) = C_K^j(x^j, x^j) + C_H^j(x^j, y^j) \neq C_K^j(y^j, y^j) + C_F^j(x^j, y^j) = C^j(y^j, x^j)$$

when $x^j \neq y^j$. Therefore, they are neither quasimetrics nor metrics.

Fix any $j \in J$. It is not difficult to check that $C^j(x^j, y^j)$ is a w**-distance** provided that the costs to hire, keep again, and fire functions c_H^j, c_K^j and c_F^j are continuous and satisfy

$$\inf_{a \in \mathbb{R}} \int_a^{a+r} c_H^j(t)dt > 0,\ \inf_{a \in \mathbb{R}} \int_a^{a+r} c_F^j(t)dt > 0,\ \text{and}\ \inf_{a \in \mathbb{R}} \int_a^{a+r} c_K^j(t)dt > 0.$$

The reader is referred to [14, Example 6] for a detailed proof.

The inconvenience-to-change the working force type j from x^j to y^j, denoted by $I^j(x^j, y^j)$, is defined by

$$I^j(x^j, y^j) \ = \ C^j(x^j, y^j) - C^j(x^j, x^j)$$

$$= \begin{cases} C_H^j(x^j, y^j) + C_K^j(x^j, x^j) - C_K^j(x^j, x^j) = \displaystyle\int_{x^j}^{y^j} c_H^j(t)dt & \text{if } y^j \geq x^j \\ C_F^j(x^j, y^j) + C_K^j(y^j, y^j) - C_K^j(x^j, x^j) = \displaystyle\int_{y^j}^{x^j} \left[c_F^j - c_K^j\right](t)dt & \text{if } y^j \leq x^j. \end{cases}$$

By Proposition 3.4 in this paper, the inconvenience-to-change the working force of type j functions, C^j, for $j \in J$ are quasimetrics.

Psychological inertia adds a fixed cost to accept to change rather than to stay denoted by $e(x) > 0$. This allows us to consider resistance-to-change as a w-distance which can be strictly positive for some x and thus is not a quasimetric. This formulation means that a change $x \curvearrowright y$ is worthwhile if the motivation-to-change $M(x, y) = U[A(x, y)] := A(x, y)$ is higher than the inconvenience-to-change plus some fixed psychological cost to accept to change, i.e. $A(x, y) \geq I(x, y) + e(x)$. For a nice paper on different aspects of psychological inertia, see [28]. Let us show that if the inconvenience-to-change $I(x, y) \geq 0$ for all $x, y \in X$ is a w-distance and if the psychological inertia term $e(x)$ is positive, then the resistance-to-change

$R(x,y) = I(x,y) + e(x)$ is also a w-distance such that $R(x,x) = e(x) > 0$ for some $x \in X$. It is sufficient to check the validity of three conditions (w1), (w2) and (w3) in Definition 3.

(w1) For all $x, y, z \in X$, $R(x,z) = I(x,z) + e(x) \leq I(x,y) + e(x) + I(y,z) + e(y) = R(x,y) + R(y,z)$ holds since $I(x,z) \leq I(x,y) + I(y,z)$.
(w2) For any $x \in X$, $R(x,\cdot) = I(x,\cdot) + e(x)$ is lower semicontinuous with respect to the second variable y because of that property of $I(x,\cdot)$.
(w3) For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $R(x,y) \leq \delta$ and $R(y,z) \leq \delta$, then $d(x,z) \leq \varepsilon$. This is true since $R(x,y) \geq I(x,y)$ and I satisfies (w3).

In summary, the model of the formation and breaking routines ensures that it is essential to extend EVP from metrics to generalized distances.

Finally, we describe the role of generalized distance versions of EVP and CFPT in the model of the formation and breaking of routines relative to hiring and firing workers. The reader is referred to [5, 6] for a unified framework in Behavioral Sciences; see also [3] for applications in Psychology, [4] for applications to capability theory of well-being, [38] for a numerical method in routine's formation with resistance to change, following worthwhile changes.

The maximization formulation of EVP considers a left-complete and left-Hausdorff quasimetric space (X, q), a weak τ-distance $p : X \times X \to \mathbb{R}_+$, a increasingly left-upper-semicontinous, not identically $-\infty$, and bounded from above payoff-to-be-increased function $g : X \to \mathbb{R}$, and initial conditions $\lambda > 0$, and $x_0 \in X$ such that $g(x) - g(x_0) \geq \lambda p(x_0, x)$ for some $x \in X$. Then, there exists $x_* \in X$ such that

(a′) $g(x_*) - g(x_0) \geq \lambda p(x_0, x_*)$;

(b′) $g(x) - g(x_*) < \lambda p(x_*, x)$ for all $x \neq x_*$;

(c′) $p(x_0, x_*) \leq \frac{\varepsilon}{\lambda}$ provided that $g(x_0) \geq \sup_{x\in X} g(x) - \varepsilon$ for some $\varepsilon > 0$.

Let us discuss here how the variational rationality approach interprets this Ekeland's result.

(a′) There exists an acceptable one step transition from an initial position to an end $x_* \in W_\lambda(x_0)$. This means that it is worthwhile to move directly from x_0 to x_*. The proof also shows that x_* is an aspiration point, i.e. $x_* \in W_\lambda(x_n)$ for all $n \in \mathbb{N}$. This means that it is worthwhile to reach x_* starting from each x_n for $n \in \mathbb{N}$.
(b′) The end is a stable position (a stationary trap): $W_\lambda(x_*) \subset \{x_*\}$. In other words, being at x_*, it is not worthwhile to move from x_* to any different action $x \neq x_*$.
(c′) The end can be reached in a feasible way: $C(x_0, x_*) = p(x_0, x_*) \leq \varepsilon/\lambda$. Then, if the agent cannot spend more than $\overline{C} > 0$ in terms of costs to move from x_0 to x_*, the agent must choose his/her acceptability ratio λ to satisfy $\varepsilon/\lambda \leq \overline{C}$. The conditions $0 \leq f(x_0) - \underline{f} < \varepsilon$ or $g(x_0) > \overline{g} - \varepsilon$ tell us that the gap between the initial and final unsatisfactions, or the gap between the maximum and initial profit is less than $\varepsilon > 0$, respectively, where $\overline{g} := \sup_{x\in X} g(x)$ and $\underline{f} := \inf_{x\in X} f(x)$.

Then, given all these behavioral simplifications, EVPs tell us that, starting from some initial position (action, or state, some being or having), there exists a worthwhile transition to a final variational trap x_* (both an aspiration point and a stable point).

How hiring and firing routines form and break.
Our model explains the formation of routines (a routinization process) in terms of the convergence of an organizational worthwhile stay and move dynamic to a variational trap, seen as a permanent routine. Our example shows how hiring and firing routines form, gradually, after a lot of repetitions, in response to a more and more similar stimuli (context, environment, cue). This models fairly well the well known and concrete study of Feldman [26] on hiring and firing routines. Let us consider the main elements which favor such a routinization process to end in a routine.

(A) The repeated "habit-loop" of a routinization process. Following Duhigg, routines in [39] can be seen as a three-part "habit loop": a cue (stimuli), a behavior (action/response) and a reward (payoff). In our model, each current period $n+1$, the cue is the stimuli given by the last unsatisfaction to do not have succeeded to reach the optimum $\overline{g} - g(x_n) = f(x_n) > 0$, the behavior is the collective hiring, firing and repeated employment action x_{n+1} and the payoff is the difference between motivation and resistance to change

$$\Delta_\lambda(x_n, x_{n+1}) = M(x_n, x_{n+1}) - \lambda R(x_n, x_{n+1}) = A(x_n, x_{n+1}) - \lambda I(x_n, x_{n+1}).$$

(B) The psychological state of the entrepreneur and his rationality. In Psychology, this state describes how, each period, the entrepreneur self regulates his activity, i.e. how, each current period, he chooses his current goal (goal setting), how he tries to reach this goal (goal striving), and how he pursues or abandons this current goal (goal pursuit or goal disengagement). Using the machinery of the VR approach in [5, 6], the present paper supposes that the current proximal goal of the entrepreneur is, each current period $n+1$, to find a worthwhile change $x_{n+1} \in W_\lambda(x_n)$. Then, the agent is proactive, bounded and procedural rational. Each period, he tries to satisfy with not too much sacrifices, balancing motivation and resistance to change, desirability and feasibility issues. This generalizes in several directions the famous static satisficing approach by Simon [40], adding, in a dynamic context, motivation (desires) and resistance to change (sacrifices) to the analysis. Worthwhile to change conditions drive, implicitly, three main and famous psychological theories of the entrepreneur: i) in Economics, the Schumpeterian theory of the entrepreneur [41] shows how the entrepreneur, being resilient to the accumulation of obstacles to change have the energy to break resistances and the motivation to innovate, ii) in Psychology, Ajzen's theory of planned behavior [42] shows how intentions to act balance perceptions of personal attractiveness (desirability), social norms, and feasibility, iii) in Management Sciences, Shapero's theory of entrepreneurial [43] intentions balances perceptions of personal desirability, feasibility, and propensity to act.

(C) The role of the environment/context. In Behavioral Sciences the stability of the context (the recurrence of a similar context/environment which acts as a trigger) is a necessary condition for the formation of routines. In this paper, the stability of the context is a condition for the convergence to a variational trap. This stability is defined by the regularity of the (VR) structure, i.e. the regularity (upper semicontinuity) of the payoff to be increased, in our model the current profit of the entrepreneur $g(.)$, and the lower semicontinuity in the second variable of the inconvenience-to-change function which is a w-distance.

(D) The routinization process. The hiring/firing process becomes more and more similar to end in an employment routine where no worker is hired or fired. Then, the structure of competences remains the same and the firm stops to innovate. The process ends in a variational trap. In our (VR) model, the formation of a hiring and firing routine comes from the balance between motivation and resistance to change, when, at the end, resistance to change wins. Routines break in the opposite case, when the habit loop breaks because motivation to change wins.

Conclusions

— The main results of this paper says that many recent versions of Ekeland's variational principle with generalized distances are equivalent to the original one in the metric setting.

— We formulate both weak τ-distance and quasi-distance versions of Ekeland's variational principle as well as Caristi's fixed-point theorem and prove that they are equivalent in the sense that one can be derived from another.

— While they are equivalent, the generalized distance versions are essential for applications in the formation and breaking routines of hiring and firing workers.

— In [22], the authors introduced the notion of λ-spaces which is much weaker than cone metric spaces and then established some critical point theorems and Ekeland-type variational theorems in the setting of λ-spaces. In [44], a modification of the notion of a w-distance was presented to further extend some fixed-point results for generalized contractive set-valued maps on complete preordered quasi-metric spaces. In [45], new fixed-point theorems under c-distance in ordered cone metric spaces were established. Our further research will examize whether they are equivalent to our results as well.

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