Taylor expansion, β-reduction and normalization

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Abstract

We introduce a notion of reduction on resource vectors, *i.e.* infinite linear combinations of resource λ-terms. The latter form the multilinear fragment of the differential λ-calculus introduced by Ehrhard and Regnier, and resource vectors are the target of the Taylor expansion of λ-terms. We show that the reduction of resource vectors contains the image, through Taylor expansion, of β-reduction in the algebraic λ-calculus, *i.e.* λ-calculus extended with weighted sums: in particular, Taylor expansion and normalization commute. We moreover exhibit a class of algebraic λ-terms, having a normalizable Taylor expansion, subsuming both arbitrary pure λ-terms, and normalizable algebraic λ-terms. For these, we prove the commutation of Taylor expansion and normalization in a more denotational sense, mimicking the Böhm tree construction.

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1 Introduction

Quantitative semantics was first proposed by Girard [15] as an alternative to domains and continuous functionals, for defining denotational models of λ-calculi with a natural interpretation of non-determinism: a type is given by a collection of "atomic states"; a term of type A is then represented by a vector (*i.e.* a possibly infinite formal linear combination) of states. The main matter is the treatment of the function space: the construction requires the interpretation of function terms to be analytic, *i.e.* defined by power series. This interpretation of λ-terms was at the origin of linear logic: the model of coherence spaces was introduced as a simplified, qualitative version of quantitative semantics [14, Appendix C].

Dealing with power series, quantitative semantics must account for infinite sums. Girard's original model can be seen as a special case of Joyal's analytic functors [17]: in particular, coefficients are sets and infinite sums are given by coproducts. This allows to give an interpretation to fixed point operators and to the pure, untyped λ-calculus. On the other hand, it does not provide a natural way to deal with weighted (*e.g.*, probabilistic) non-determinism, where coefficients are taken in an external semiring of scalars.

In the early 2000's, Ehrhard introduced alternative presentations of quantitative semantics [8, 9], limited to a typed setting, but where types can be interpreted as vector spaces, or more generally semimodules over an arbitrary fixed semiring; these spaces are moreover equipped with a linear topology, allowing to interpret proofs as linear and continuous maps, in a standard sense. In this setting, the formal differentiation of power series recovers its usual meaning of linear approximation of a function, and the Taylor expansion formula holds: if ϕ is analytic then $\phi(\alpha) = \sum_{n\in\mathbf{N}} \frac{1}{n!}(\phi^{(n)}(0)) \cdot \alpha^n$ where $\phi^{(n)}(0)$ is the n-th derivative of ϕ computed at 0, which is an n-linear map, and α^n is the n-th tensor power of α.

Ehrhard and Regnier gave a computational meaning to such derivatives by introducing linearized variants of application and substitution in the λ-calculus, which led to the differential λ-calculus [11], and then the resource λ-calculus [13] — the latter retains iterated

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derivatives at zero as the only form of application. They were then able to recast the above Taylor expansion formula in a syntactic, untyped setting: to every λ-term M, they associate a vector $\Theta(M)$ of resource λ-terms, *i.e.* terms of the resource λ-calculus.

The Taylor expansion of a λ-term can be seen as an intermediate, infinite object, between the term and its denotation in quantitative semantics. Indeed, resource terms still retain a dynamics, even if a very simple, finitary one: the size of terms is strictly decreasing under reduction. Furthermore, normal resource terms are in close relationship with the atomic states of quantitative semantics of the pure λ-calculus (or equivalently with the elements of a reflexive object in the relational model [3]; or with normal type derivations in a non-idempotent intersection type system [5]). So the normal form of $\Theta(M)$ should coincide with the denotation of M, which allows for a very generic description of quantitative semantics.

Other approaches to quantitative semantics impose a constraint *a priori*, either on the computational model or on the treatment of scalar coefficients: *e.g.*, the denotational model of finiteness spaces [9] is, by design, limited to strongly normalizing computation; probabilistic coherence spaces [4] can accommodate fixpoints but are limited to probabilistic distributions of terms; weighted relational models [20, 19] rely on a completeness property of the semiring; and Girard's original quantitative semantics [15] used sets as coefficients.

The "normalization of Taylor expansion" approach is more permissive. For instance, we can consider the algebraic λ-calculus [24], a very generic, non-uniform extension of λ-calculus, augmenting the syntax with formal finite linear combinations of terms: the Taylor expansion operator extends naturally to this setting. Of course, there is a price attached to such liberality: in general, the normal form of a vector of resource λ-terms is not well defined; and it is not difficult to find algebraic λ-terms whose Taylor expansion is not normalizable.

Ehrhard and Regnier proved, however, that the Taylor expansion of a pure λ-term M is always normalizable [13]. This can be seen as a new proof of the fact that Girard's quantitative semantics of pure λ-terms uses finite cardinals only [16]. They moreover established that this normal form is exactly the Taylor expansion of the Böhm tree $\mathsf{BT}(M)$ of M [12]. Both results rely heavily on the uniformity of the pure λ-calculus: all resource terms in $\Theta(M)$ follow a single tree pattern. This is a bit disappointing since quantitative semantics was introduced as a model of non-determinism, which is incompatible with uniformity.

In the present paper, we transpose Ehrhard and Regnier's results to the full algebraic λ-calculus, which requires the development of a new proof technique, not relying on uniformity:

1. we prove that normalizable algebraic λ-terms have a normalizable Taylor expansion (Theorem 47, Section 7);
2. we introduce a notion of reduction on vectors of resource λ-terms (Section 5), containing the image of β-reduction on algebraic λ-terms by Taylor expansion (Section 6);
3. we deduce that, for normalizable algebraic λ-terms, normalization and Taylor expansion commute on the nose (Section 7);
4. we introduce the class of *hereditarily determinable* terms, containing in particular all pure λ-terms as well as all normalizable algebraic λ-terms, together with a notion of finite normal form approximants, called *d-determinate forms*; and we prove that if M is hereditarily determinable, then $\Theta(M)$ is normalizable, and its normal form is entirely characterized by the d-determinate forms of M (Section 8).

Step 1 is a generalization of previous work by Ehrhard [10] in a typed setting, and of our own work with Pagani and Tasson [22] for strongly normalizable terms. Let us stress that, in those previous contributions, the actual commutation of normalization and Taylor expansion was never considered, let alone any analogue of step 4. The results of Ehrhard and Regnier about the pure λ-calculus [13, 12] provided analogues of steps 1 and 4 (which entails 3), using uniformity to skip step 2. When restricted to pure λ-terms, our results actually provide a new proof, using very different tools, that the normal form of $\Theta(M)$ is isomorphic to $\mathsf{BT}(M)$.

In their full extent, our results provide a generalization of the notion of non-deterministic Böhm tree [6] in a weighted, quantitative setting. This may appeal to anyone with an interest in denotational semantics or infinitary normal forms, in a non-deterministic setting, but the techniques we rely on may be familiar only to a much narrower audience, so we start the paper by reviewing them briefly: in Section 2 we recall the notion of finiteness space and detail the case of linear-continuous maps defined by summable families of vectors, which is pervasive in the paper; we review the syntax and dynamics of the resource λ-calculus in Section 3; then we present the Taylor expansion construction and some of its basic properties in Section 4 — this includes results that might be considered folklore by some, but for which we found no previously published reference.

2 Finiteness spaces and summable functions

Finiteness spaces were introduced by Ehrhard [9] as a model of linear logic, giving an account of quantitative semantics in a standard linear algebraic setting. In short, types are interpreted as topological vector spaces of a particular form, linear logic proofs are interpreted as linear-continuous maps, and λ-terms as analytic maps, subject to Taylor expansion.

For the rest of the paper, $\mathbf{S}$ will denote an arbitrary commutative semiring $\mathbf{S} = (\underline{\mathbf{S}}, 0_{\mathbf{S}}, +_{\mathbf{S}}, 1_{\mathbf{S}}, \cdot_{\mathbf{S}})$. We will often abuse notation and identify $\mathbf{S}$ with its carrier set $\underline{\mathbf{S}}$; we may also omit the subscripts on $+$, $\cdot$, 0 and 1, and write ab for $a \cdot b$. Given a set X, we write $\mathbf{S}^X$ for the semimodule of formal linear combinations of elements in X: a *vector* $\xi \in \mathbf{S}^X$ is nothing but an X-indexed family of scalars $(\xi_x)_{x\in X}$, that we may also denote by $\sum_{x\in X} \xi_x.x$. The *support* $|\xi|$ of a vector $\xi \in \mathbf{S}^X$ is the set of elements of X having a non-zero coefficient in ξ: $|\xi| := \{x \in X \; ; \; \xi_x \neq 0\}$. We write $\mathbf{S}[X]$ for the set of vectors with finite support.

If X is a set, we write $\mathfrak{P}(X)$ for the powerset of X. We call *structure* on X any $\mathfrak{X} \subseteq \mathfrak{P}(X)$ and then we write $\mathfrak{X}^\perp := \{x' \subseteq X \; ; \; \text{for all } x \in \mathfrak{X},\, x \cap x' \text{ is finite}\}$. A *relational finiteness space* $\mathcal{X}$ is a pair $(\|\mathcal{X}\|, \mathfrak{F}(\mathcal{X}))$, where $\|\mathcal{X}\|$ is a set (the *web* of $\mathcal{X}$) and $\mathfrak{F}(\mathcal{X})$ is a structure on $\|\mathcal{X}\|$ such that $\mathfrak{F}(\mathcal{X}) = \mathfrak{F}(\mathcal{X})^{\perp\perp}$: $\mathfrak{F}(\mathcal{X})$ is called the *finiteness structure* of $\mathcal{X}$, and we say $x \subseteq \|\mathcal{X}\|$ is *finitary* in $\mathcal{X}$ iff $x \in \mathfrak{F}(\mathcal{X})$. The *finiteness space* generated by $\mathcal{X}$ is the set $\mathbf{S}\langle\mathcal{X}\rangle$ of vectors on $\|\mathcal{X}\|$ with finitary support: $\xi \in \mathbf{S}\langle\mathcal{X}\rangle$ iff $|\xi| \in \mathfrak{F}(\mathcal{X})$. Finitary subsets are downwards closed for inclusion, and finite unions of finitary subsets are finitary, hence $\mathbf{S}\langle\mathcal{X}\rangle$ is a subsemimodule of $\mathbf{S}^X$. The least (resp. greatest) finiteness structure on X is the set $\mathfrak{P}_f(X)$ of finite subsets of X (resp. $\mathfrak{P}(X)$), generating the finiteness space $\mathbf{S}[X]$ (resp. $\mathbf{S}^X$).

We do not describe the whole category of finiteness spaces and linear-continuous maps here. In particular we do not recall the details of the linear topology induced on $\mathbf{S}\langle\mathcal{X}\rangle$ by $\mathfrak{F}(\mathcal{X})$: the reader may refer to Ehrhard's original paper. In the remaining of this section, we focus on a very particular case, where the finiteness structure on base types is trivial: linear-continuous maps are then univocally generated by summable functions.

▶ **Definition 1.** Let $\overrightarrow{\xi} = (\xi_i)_{i\in I} \in (\mathbf{S}^X)^I$ be a family of vectors: write $\xi_i = \sum_{x\in X} \xi_{i,x}.x$. We say $\overrightarrow{\xi}$ is *summable* if, for all $x \in X$, $\{i \in I \; ; \; x \in |\xi_i|\}$ is finite. In this case, we define the sum $\sum \overrightarrow{\xi} = \sum_{i\in I} \xi_i \in \mathbf{S}^X$ in the obvious, pointwise way: $(\sum \overrightarrow{\xi})_x := \sum_{i\in I} \xi_{i,x}$.

Of course, any finite family of vectors is summable and we use standard notations for finite sums. The reader can moreover check that the family $\overrightarrow{\xi}$ is summable iff the set $\{(i, x) \in I \times X \; ; \; \xi_{i,x} \neq 0\}$ is finitary in the relational arrow finiteness space $(I, \mathfrak{P}(I)) \multimap (X, \mathfrak{P}(X))$

as defined by Ehrhard [9, in particular Lemma 3]. In particular, summable families form a finiteness space, hence an $\mathbf{S}$-semimodule. If $f : X \to \mathbf{S}^Y$ is a summable function (*i.e.* the family $(f(x))_{x\in X}$ is summable) and $\xi \in \mathbf{S}^X$, we write $f \cdot \xi := \sum_{x\in X} \xi_x.f(x)$.

▶ **Definition 2.** Let $\phi : \mathbf{S}^X \to \mathbf{S}^Y$. We say ϕ is *linear-continuous* if, for all summable family $(\xi_i)_{i\in I} \in (\mathbf{S}^X)^I$, the family $(\phi(\xi_i))_{i\in I}$ is summable and, for all family of scalars, $(a_i)_{i\in I} \in \mathbf{S}^I$, we have $\phi(\sum_{i\in I} a_i.\xi_i) = \sum_{i\in I} a_i.\phi(\xi_i)$.

One might check that a map $\phi : \mathbf{S}^X \to \mathbf{S}^Y$ is linear-continuous in the sense of Definition 2 iff it is linear and continuous in the sense of the linear topology of finiteness spaces, observing that the topology on $\mathbf{S}^X = \mathbf{S}\langle(X, \mathfrak{P}(X))\rangle$ is the product topology [9, Section 3]. As a general fact in finiteness spaces, linear-continuous maps are those defined by finitary matrices:

▶ **Lemma 3.** *If $\phi : \mathbf{S}^X \to \mathbf{S}^Y$ is linear-continuous then its restriction $\phi|_X : X \to \mathbf{S}^Y$ is summable and $\phi(\xi) = \phi|_X \cdot \xi$. Conversely, if $f : X \to \mathbf{S}^Y$ is summable then $\xi \mapsto f \cdot \xi$ is linear-continuous.*

It is easy to generalize Definitions 1 and 2 to n-ary functions. Following the constructions of finiteness spaces, n-ary summable functions $f : X_1 \times \cdots \times X_n \to \mathbf{S}^Y$ correspond with n-linear-hypocontinuous maps [9, Section 3] from $\mathbf{S}^{X_1} \times \cdots \times \mathbf{S}^{X_n}$ to $\mathbf{S}^Y$, rather than the more restrictive multilinear-continuous maps. In the simpler setting of summable functions, though, both notions coincide, since $\mathbf{S}^X$ is always locally linearly compact [9, Proposition 15], so the previous lemma still holds for n-ary functions. In the remaining, we will thus identify n-ary summable functions with n-linear-continuous maps.

3 The resource λ-calculus

We fix an infinite, denumerable set $\mathcal{V}$ of variables, that we denote by small letters x, y, z. We define the sets Δ of *resource terms* and $!\Delta$ of *resource monomials* simultaneously as follows:[1]

$$\Delta \ni s, t, u, v, w ::= x \mid \lambda x\, s \mid \langle s\rangle\ \overline{t} \qquad \text{and} \qquad !\Delta \ni \overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w} ::= [] \mid [s] \cdot \overline{t}.$$

Terms are considered up to α-equivalence and monomials up to permutativity: we write $[t_1, \dots, t_n]$ for $[t_1]\cdot(\cdots([t_n]\cdot[]))$ and equate $[t_1, \dots, t_n]$ with $[t_{f(1)}, \dots, t_{f(n)}]$ for all permutation f of $\{1, \dots, n\}$, so that resource monomials coincide with finite multisets of resource terms.[2] We will then write $\overline{s} \cdot \overline{t}$ for the multiset union of $\overline{s}$ and $\overline{t}$, and $\#[s_1, \dots, s_n] := n$.

We call resource expression any resource term or resource monomial and write $(!)\Delta$ for either Δ or $!\Delta$: whenever we use this notation several times in the same context, all occurrences consistently denote the same set.

▶ **Definition 4.** We define by induction over a resource expression $e \in (!)\Delta$, its *size* $\mathbf{s}(e) \in \mathbf{N}$ and its *height* $\mathbf{h}(e) \in \mathbf{N}$, by setting $\mathbf{s}(x) := 1$ and $\mathbf{h}(x) := 1$ and, supposing $\overline{t} = [t_1, \dots, t_n]$:

$$\mathbf{s}(\lambda x\, s) := 1 + \mathbf{s}(s) \quad \mathbf{s}(\langle s\rangle\ \overline{t}) := 1 + \mathbf{s}(s) + \mathbf{s}(\overline{t}) \quad \mathbf{s}(\overline{t}) := \mathbf{s}(t_1) + \cdots + \mathbf{s}(t_n)$$
$$\mathbf{h}(\lambda x\, s) := 1 + \mathbf{h}(s) \quad \mathbf{h}(\langle s\rangle\ \overline{t}) := \max(\mathbf{h}(s), 1 + \mathbf{h}(\overline{t})) \quad \mathbf{h}(\overline{t}) := \max(\mathbf{h}(t_1), \dots, \mathbf{h}(t_n)).$$

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[1] We use a variant of BNF notation: *e.g.*, $!\Delta$ is inductively generated by $[]$ and addition of a resource term to a monomial; and we denote monomials by overlined letters $\overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}$, possibly with sub- and superscripts.

[2] Resource monomials are also called *bags*, *bunches* or *poly-terms* in the literature, but we prefer to strengthen the analogy with power series here.

$$\frac{}{\langle \lambda x\, s\rangle\ \overline{t} \to_\partial \partial_x s \cdot \overline{t}} \qquad \frac{s \to_\partial \sigma'}{\lambda x\, s \to_\partial \lambda x\, \sigma'} \qquad \frac{s \to_\partial \sigma'}{\langle s\rangle\ \overline{t} \to_\partial \langle \sigma'\rangle\ \overline{t}} \qquad \frac{\overline{t} \to_\partial \overline{\tau}'}{\langle s\rangle\ \overline{t} \to_\partial \langle s\rangle\ \overline{\tau}'} \qquad \frac{s \to_\partial \sigma'}{[s]\cdot \overline{t} \to_\partial [\sigma']\cdot \overline{t}}$$

Figure 1 Rules of resource reduction $\to_\partial$.

In particular $\mathbf{s}(s) > 0$ and $\mathbf{h}(s) > 0$ for all $s \in \Delta$, and $\mathbf{s}(\overline{s}) \geq \#\overline{s}$ for all $\overline{s} \in !\Delta$. For all $e \in (!)\Delta$, we write $\mathsf{fv}(e)$ for the set of its free variables and $\mathbf{n}_x(e) \in \mathbf{N}$ for the number of free occurrences of x in e. It should be clear that $\mathbf{n}_x(e) \leq \mathbf{s}(e)$, and $x \in \mathsf{fv}(e)$ iff $\mathbf{n}_x(e) \neq 0$.

In the resource λ-calculus, the substitution $e[s/x]$ of a term s for a variable x in e admits a linear counterpart: this operator was initially introduced in the differential λ-calculus [11] in the form of a partial differentiation operation, reflecting the interpretation of λ-terms in quantitative semantics. Here we only consider those derivatives involved in the Taylor expansion of λ-terms: iterated derivatives at 0, *e.g.* $(\frac{\partial^n e}{\partial x^n} \cdot (u_1, \dots, u_n))[0/x]$, that are n-linear symmetric in the u_i's, and that we denote more concisely by $\partial_x e \cdot [u_1, \dots, u_n]$.

To introduce this construct, we need to consider formal finite sums of resource expressions, and to extend all syntactic constructs by linearity: if $\sigma = \sum_{i=1}^n s_i \in \mathbf{N}[\Delta]$ and $\overline{\tau} = \sum_{j=1}^p \overline{t}_j \in \mathbf{N}[!\Delta]$, we set $\lambda x\, \sigma := \sum_{i=1}^n \lambda x\, s_i$, $\langle \sigma\rangle\ \overline{\tau} := \sum_{i=1}^n \sum_{j=1}^p \langle s_i\rangle\ \overline{t}_j$ and $[\sigma]\cdot\overline{\tau} := \sum_{i=1}^n \sum_{j=1}^p [s_i]\cdot\overline{t}_j$.

▶ **Definition 5.** If $\overline{u} = [u_1, \dots, u_n] \in !\Delta$ and $1 \leq i_1 < \cdots < i_l \leq n$, we write $\overline{u}_{\{i_1,\dots,i_l\}} := [u_{i_1}, \dots, u_{i_l}]$. Then we define the *$n$-linear substitution* of $\overline{u}$ for x in e inductively as follows: [3]

$$\partial_x y \cdot \overline{u} := \begin{cases} y & \text{if } y \neq x \text{ and } n = 0 \\ u_1 & \text{if } y = x \text{ and } n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \partial_x \langle s\rangle\ \overline{t} \cdot \overline{u} := \sum_{(I_0, I_1) \text{ partition of } \{1,\dots,n\}} \langle \partial_x s \cdot \overline{u}_{I_0}\rangle\ \partial_x \overline{t} \cdot \overline{u}_{I_1}$$

$$\partial_x \lambda y\, s \cdot \overline{u} := \lambda y\ (\partial_x s \cdot \overline{u}) \qquad \partial_x [s_1, \dots, s_k] \cdot \overline{u} := \sum_{(I_1, \dots, I_k) \text{ partition of } \{1,\dots,n\}} [\partial_x s_1 \cdot \overline{u}_{I_1}, \dots, \partial_x s_k \cdot \overline{u}_{I_k}].$$

▶ **Lemma 6.** *If $e' \in |\partial_x e \cdot \overline{u}|$ then $\mathbf{n}_x(e) = \#\overline{u}$ and $\mathsf{fv}(e') = (\mathsf{fv}(e) \setminus \{x\}) \cup \mathsf{fv}(\overline{u})$. Moreover, we have $\mathbf{s}(e') = \mathbf{s}(e) + \mathbf{s}(\overline{u}) - \#\overline{u}$ and $\mathbf{h}(e) \leq \mathbf{h}(e') \leq \mathbf{h}(e) + \mathbf{h}(\overline{u}) - 1$.*

If $\to$ is a reduction relation, we write $\to^?$ (resp. $\to^*$) for its reflexive (resp. reflexive and transitive) closure. A redex is a term of the form $\langle \lambda x\, t\rangle\ \overline{u}$, and its reduct is $\partial_x t \cdot \overline{u} \in \mathbf{N}[\Delta]$.

▶ **Definition 7.** We define the *resource reduction* relation $\to_\partial \subseteq (!)\Delta \times \mathbf{N}[(!)\Delta]$ inductively by the rules of Fig. 1. We then extend $\to_\partial$ to finite sums, setting $\epsilon \to_\partial \epsilon'$ if $\epsilon = \sum_{i=0}^n e_i$ and $\epsilon' = \sum_{i=0}^n \epsilon'_i$ with $e_0 \to_\partial \epsilon'_0$ and, for all $i \geq 1$, $e_i \to_\partial^? \epsilon'_i$.

The way the reduction is extended to sums is sufficient [13] to ensure the strong confluence of $\to_\partial^?$. Moreover, the effect of reduction on the size of terms is very regular:

▶ **Lemma 8.** *If $e \to_\partial \epsilon'$ and $e' \in |\epsilon'|$ then $\mathsf{fv}(e') = \mathsf{fv}(e)$ and $\mathbf{s}(e') + 2 \leq \mathbf{s}(e) \leq 2\mathbf{s}(e') + 2$.*

We will write $e \geq_\partial e'$ if $e \to_\partial^* \epsilon'$ with $e' \in |\epsilon'|$. We obtain that $\{e' \ ; \ e \geq_\partial e'\}$ is finite hence $\to_\partial$ is strongly normalizing: we write $\mathsf{NF}(\epsilon)$ for the normal form of ϵ.

4 Resource vectors and Taylor expansion of algebraic λ-terms

In order to write down the Taylor expansion formula, we will need inverses of natural numbers to be available: say $\mathbf{S}$ *has fractions* if there is a semiring morphism from $\mathbf{Q}^+$ (the semiring

[3] We adhere to Barendregt's convention and, *e.g.*, implicitly assume $y \notin \mathsf{fv}(\overline{u})$ in the abstraction case.

of non negative rational numbers) to $\mathbf{S}$. We abuse notation and identify rationals with their images; moreover, we silently assume $\mathbf{S}$ has fractions whenever we use them.

4.1 Resource vectors

A vector $\sigma = \sum_{s\in\Delta} \sigma_s.s$ of resource terms will be called a *term vector*. Similarly, we will call *monomial vector* any vector of resource monomials. A *resource vector* will be any of a term vector or a monomial vector and, again, we will write $\mathbf{S}^{(!)\Delta}$ for either $\mathbf{S}^{\Delta}$ or $\mathbf{S}^{!\Delta}$. The syntactic constructs are extended to resource vectors by linearity: for all $\sigma \in \mathbf{S}^{\Delta}$ and $\overline{\sigma}, \overline{\tau} \in \mathbf{S}^{!\Delta}$, we set $\lambda x\, \sigma := \sum_{s\in\Delta} \sigma_s.\lambda x\, s$, $\langle\sigma\rangle\ \overline{\tau} := \sum_{s\in\Delta,\overline{t}\in!\Delta} \sigma_s\overline{\tau}_{\overline{t}}.\ \langle s\rangle\ \overline{t}$, $[\sigma] := \sum_{s\in\Delta}\sigma_s.[s]$ and $\overline{\sigma}\cdot\overline{\tau} := \sum_{\overline{s},\overline{t}\in!\Delta} \overline{\sigma}_{\overline{s}}\overline{\tau}_{\overline{t}}.\overline{s}\cdot\overline{t}$. For these sums to be meaningful, we need to prove that the constructors of the calculus define summable functions, which is quite straightforward: *e.g.*, for every $\overline{u} \in !\Delta$ there is at most one s with $\overline{u} \in |[s]|$ (*i.e.* $\overline{u} = [s]$); and there are finitely many pairs $(\overline{s}, \overline{t})$ with $\overline{u} \in |\overline{s}\cdot\overline{t}|$ (*i.e.* $\overline{u} = \overline{s}\cdot\overline{t}$). Similarly, for linear substitution:

▶ **Lemma 9.** *The family $(\partial_x e\cdot\overline{s})_{(e,\overline{s})\in(!)\Delta\times!\Delta}$ is summable.*

Proof. First observe that we can consider $\partial_x e\cdot\overline{s} \in \mathbf{N}[(!)\Delta] \subseteq \mathbf{S}^{(!)\Delta}$ via the unique semiring morphism from $\mathbf{N}$ into $\mathbf{S}$. By Lemma 6, if $e' \in |\partial_x e\cdot\overline{s}|$ then $\mathsf{fv}(e) \subseteq \mathsf{fv}(e')\cup\{x\}$, $\mathsf{fv}(\overline{s}) \subseteq \mathsf{fv}(e')$, $\mathbf{s}(e) \le \mathbf{s}(e')$ and $\mathbf{s}(\overline{s}) \le \mathbf{s}(e')$: e' being fixed, there are finitely many such pairs $(e, \overline{s})$. ◀

We thus obtain a bilinear-continuous map: $\partial_x\epsilon\cdot\overline{\sigma} := \sum_{e\in(!)\Delta,\overline{s}\in!\Delta} \epsilon_e\overline{\sigma}_{\overline{s}}.\partial_x e\cdot\overline{s}$. By contrast, the usual substitution is not linear, so it must be defined directly for resource vectors.

▶ **Definition 10.** We define the substitution $e[\sigma/x] \in \mathbf{S}^{(!)\Delta}$ of $\sigma \in \mathbf{S}^{\Delta}$ for a variable x in $e \in (!)\Delta$ inductively as follows (assuming $x \neq y$ and $\overline{t} = [t_1,\dots,t_n]$):

$$x[\sigma/x] := \sigma \qquad (\lambda y\, s)[\sigma/x] := \lambda y\, s[\sigma/x]$$
$$y[\sigma/x] := y \qquad (\langle s\rangle\ \overline{t})[\sigma/x] := \langle s[\sigma/x]\rangle\ \overline{t}[\sigma/x] \qquad \overline{t}[\sigma/x] := [t_1[\sigma/x],\dots,t_n[\sigma/x]].$$

▶ **Lemma 11.** *For all $e \in (!)\Delta$, $x \in \mathcal{V}$ and $\sigma \in \mathbf{S}^{\Delta}$: if $x \notin \mathsf{fv}(e)$ then $e[\sigma/x] = e$; otherwise, if $x \in \mathsf{fv}(e)$ then $e[0/x] = 0$. Moreover, for all $e' \in |e[\sigma/x]|$, $\mathsf{fv}(e) \subseteq \mathsf{fv}(e')\cup\{x\}$, $\mathsf{fv}(e') \subseteq \mathsf{fv}(e)\cup\mathsf{fv}(\sigma)$ and $\mathbf{s}(e') \ge \mathbf{s}(e)$.*

A consequence of the previous lemma is that the function $e \in (!)\Delta \mapsto e[\sigma/x] \in \mathbf{S}^{(!)\Delta}$ is summable: we thus write $\epsilon[\sigma/x] := \sum_{e\in\mathbf{S}^{(!)\Delta}} \epsilon_e.e[\sigma/x]$.

4.2 Promotion

For all $\sigma \in \mathbf{S}^{\Delta}$, we define $\sigma^n \in \mathbf{S}^{!\Delta}$ by induction on $n \in \mathbf{N}$: $\sigma^0 = []$ and $\sigma^{n+1} = [\sigma]\cdot\sigma^n$. Observe that the family $(\sigma^n)_{n\in\mathbf{N}}$ of monomial vectors is summable because the supports $|\sigma^n|$ for $n \in \mathbf{N}$ are pairwise disjoint. We then define the promotion of σ as $\sigma^! := \sum_{n\in\mathbf{N}} \frac{1}{n!}\sigma^n$.

▶ **Lemma 12.** *For all σ and $\tau \in \mathbf{S}^{\Delta}$, $\sigma^![\tau/x] = (\sigma[\tau/x])^!$.*

Proof. By the linear-continuity of $\epsilon \mapsto \epsilon[\sigma/x]$, it is sufficient to prove that $\sigma^n[\tau/x] = (\sigma[\tau/x])^n$ which follows from the n-linear-continuity of $(\sigma_1,\dots,\sigma_n) \mapsto [\sigma_1,\dots,\sigma_n]$. ◀

For all $k, l_1,\dots,l_n \in \mathbf{N}$ such that $k = l_1+\cdots+l_n$, we write $\left[{k \atop l_1,\dots,l_n}\right] := \frac{k!}{\prod_{i=1}^n l_i!}$, which is always an integer. These *multinomial coefficients* generalize the binomial coefficients: $\binom{k}{l} = \left[{k \atop l,k-l}\right]$. They arise naturally in the description of various quantitative denotational semantics of linear logic and λ-calculus [9, 4], more specifically in the interpretation of the promotion rule. Here we will use the following result, which boils down to an exercise on the combinatorics of the partitions of $\{1,\dots,k\}$:

▶ **Lemma 13.** *If $s \in \Delta$, $\bar{t} = [t_1, \dots, t_n] \in !\Delta$, and $\rho \in \mathbf{S}^\Delta$, then*

$$\partial_x \langle s \rangle\ \bar{t} \cdot \rho^k = \sum_{l=0}^{k} [{}_{l,k-l}^{\ k}] \left\langle \partial_x s \cdot \rho^l \right\rangle \partial_x \bar{t} \cdot \rho^{k-l} \text{ and } \partial_x \bar{t} \cdot \rho^k = \sum_{\substack{l_1,\dots,l_n \in \mathbf{N} \\ l_1+\dots+l_n=k}} [{}_{l_1,\dots,l_n}^{\ \ k}][\partial_x t_1 \cdot \rho^{l_1}, \dots, \partial_x t_n \cdot \rho^{l_n}].$$

▶ **Lemma 14.** *The following identities hold (assuming $x \neq y$ and $\bar{\tau} = [\tau_1, \dots, \tau_n]$):*

$$\partial_x x \cdot \rho^! = \rho \qquad \partial_x \lambda y\, \sigma \cdot \rho^! = \lambda y \left(\partial_x \sigma \cdot \rho^!\right) \qquad \partial_x \bar{\tau} \cdot \rho^! = [\partial_x \tau_1 \cdot \rho^!, \dots, \partial_x \tau_n \cdot \rho^!].$$
$$\partial_x y \cdot \rho^! = y \qquad \partial_x \langle \sigma \rangle\ \bar{\tau} \cdot \rho^! = \left\langle \partial_x \sigma \cdot \rho^! \right\rangle \partial_x \bar{\tau} \cdot \rho^!$$

Proof. Since each syntactic constructor is multilinear-continuous and $\epsilon \mapsto \partial_x \epsilon \cdot \rho^!$ is linear-continuous for a fixed ρ, it is sufficient to consider the case of $\partial_x e \cdot \rho^!$ for a resource expression $e \in (!)\Delta$. First observe that, by Lemma 6, if $k = \mathbf{n}_x(e)$ then $\partial_x e \cdot \rho^! = \frac{1}{k!}.\partial_x e \cdot \rho^k$. In particular the case of variables is straightforward.

In the abstraction case, $\partial_x \lambda x\, s \cdot \rho^! = \lambda x \left(\partial_x s \cdot \rho^!\right)$ follows from the linear-continuity of multilinear substitution and the fact that $\partial_x \lambda x\, s \cdot \bar{t} = \lambda x \left(\partial_x s \cdot \bar{t}\right)$ for all $\bar{t} \in !\Delta$.

If $e = \langle s \rangle\ \bar{t}$, write $l = \mathbf{n}_x(s)$ and $m = \mathbf{n}_x(\bar{t})$. Then $k = l+m$ and by the previous lemma and Lemma 6 again $\partial_x e \cdot \rho^k = [{}_{l,m}^{\ k}].\left\langle \partial_x s \cdot \rho^l \right\rangle \partial_x \bar{t} \cdot \rho^m$ and then $\frac{1}{k!}.\partial_x e \cdot \rho^k = \left\langle \frac{1}{l!}.\partial_x s \cdot \rho^l \right\rangle \frac{1}{m!} \partial_x \bar{t} \cdot \rho^m$. The case of monomials is similar. ◀

▶ **Lemma 15.** *For all $\epsilon \in \mathbf{S}^{(!)\Delta}$ and $\sigma \in \mathbf{S}^\Delta$, $\epsilon[\sigma/x] = \partial_x \epsilon \cdot \sigma^!$.*

Proof. By the linear-continuity of $\epsilon \mapsto \partial_x \epsilon \cdot \sigma^!$ and $\epsilon \mapsto \epsilon[\sigma/x]$, it suffices to show $e[\sigma/x] = \partial_x e \cdot \sigma^!$ for $e \in (!)\Delta$: the proof is then by induction on e, using Lemma 14 in each case. ◀

By Lemma 12, we thus obtain $\partial_x \sigma^! \cdot \tau^! = \left(\partial_x \sigma \cdot \tau^!\right)^!$ which can be seen as a counterpart of the functoriality of promotion in linear logic. To our knowledge it is the first published proof of such a result for resource vectors. This will enable us to prove the commutation of Taylor expansion and substitution (Lemma 17), another unsurprising yet non-trivial result.

4.3 Taylor expansion of algebraic λ-terms

Since resource vectors form a module, there is no reason to restrict the source language of Taylor expansion to the pure λ-calculus. We will thus consider the following set of terms:

$$\Sigma_{\mathbf{S}} \ni M, N, P ::= x \mid \lambda x\, M \mid (M)\ N \mid 0 \mid a.M \mid M + N$$

where a ranges in $\mathbf{S}$. For now, terms are considered up to the usual α-equivalence only: the null term 0, scalar multiplication $a.M$ and sum $M + N$ are purely syntactic constructs.

▶ **Definition 16.** We define the Taylor expansion $\Theta(M) \in \mathbf{S}^{(!)\Delta}$ of $M \in \Sigma_{\mathbf{S}}$ by induction:

$$\Theta(x) := x \qquad \Theta(\lambda x\, M) := \lambda x\, \Theta(M) \qquad \Theta((M)\ N) := \langle \Theta(M) \rangle\ \Theta(N)^!$$
$$\Theta(0) := 0 \qquad \Theta(a.M) := a.\Theta(M) \qquad \Theta(M + N) := \Theta(M) + \Theta(N).$$

▶ **Lemma 17.** *For all $M, N \in \Sigma_{\mathbf{S}}$, $\Theta(M[N/x]) = \partial_x \Theta(M) \cdot \Theta(N)^! = \Theta(M)[\Theta(N)/x]$.*

Proof. By induction on M, using Lemmas 14 and 15. ◀

Write $M \simeq_\Theta N$ if $\Theta(M) = \Theta(N)$. This defines a *congruence* on terms: $\simeq_\Theta$ is an equivalence relation and $M \simeq_\Theta M'$ implies $\lambda x\, M \simeq_\Theta \lambda x\, M'$, $(M)\ N \simeq_\Theta (M')\ N$, $(N)\ M \simeq_\Theta (N)\ M'$, $a.M \simeq_\Theta a.M'$, $M + N \simeq_\Theta M' + N$ and $N + M \simeq_\Theta N + M'$. Write $\simeq_{\mathrm{v}}$ for the least

$$\begin{array}{llll}
\text{(a)} & 0+M \simeq_{\mathrm{v}} M & M+N \simeq_{\mathrm{v}} N+M & (M+N)+P \simeq_{\mathrm{v}} M+(N+P) \\
 & 0.M \simeq_{\mathrm{v}} 0 & 1.M \simeq_{\mathrm{v}} M & a.M+b.M \simeq_{\mathrm{v}} (a+b).M \\
 & a.0 \simeq_{\mathrm{v}} 0 & a.(b.M) \simeq_{\mathrm{v}} ab.M & a.(M+N) \simeq_{\mathrm{v}} a.M+a.N \\
\text{(b)} & \lambda x\, 0 \simeq_{\mathrm{v}} 0 & \lambda x\ (a.M) \simeq_{\mathrm{v}} a.\lambda x\, M & \lambda x\ (M+N) \simeq_{\mathrm{v}} \lambda x\, M+\lambda x\, N \\
 & (0)\ P \simeq_{\mathrm{v}} 0 & (a.M)\ P \simeq_{\mathrm{v}} a.(M)\ P & (M+N)\ P \simeq_{\mathrm{v}} (M)\ P+(N)\ P
\end{array}$$

Figure 2 Equations of vector λ-terms: (a) equations of $\mathbf{S}$-module; (b) linearity.

congruence containing the equations of Fig. 2, so that $M \simeq_{\mathrm{v}} N$ implies $M \simeq_\Theta N$. We call *vector λ-terms* the elements of the quotient $\Sigma_{\mathbf{S}}/{\simeq_{\mathrm{v}}}$, which forms an $\mathbf{S}$-semimodule: those are the terms of the previously studied algebraic λ-calculus [24, 1] (where they were called algebraic λ-terms; here we reserve this name for another, simpler notion).

Notice that Taylor expansion is not injective on vector λ-terms in general. For instance, if $\mathbf{S} = \mathbf{B}$ (the commutative semiring of booleans: $\underline{\mathbf{B}} = \{0,1\}$, $+_{\mathbf{B}} = \vee$ and $\cdot_{\mathbf{B}} = \wedge$), we obtain $(x)\ 0 + (x)\ x \simeq_\Theta (x)\ x$. It is moreover well known [24, 2] that β-reduction in a semimodule of terms is inconsistent in presence of negative coefficients.

▶ **Example 18.** Consider $\delta_M := \lambda x\ (M + (x)\ x)$ and $\infty_M := (\delta_M)\ \delta_M$. Observe that ∞_M β-reduces to $M + \infty_M$. Then any congruence $\simeq$ on $\Sigma_{\mathbf{Z}}$ containing β-reduction and the equations of $\mathbf{Z}$-semimodule is inconsistent: $0 \simeq \infty_M - \infty_M \simeq M + \infty_M - \infty_M \simeq M$.

It is not our purpose here to survey the various possible approaches to the rewriting theory of β-reduction on vector λ-terms: we refer the reader to the literature on algebraic λ-calculi [24, 2, 1, 7] for several proposals. Our focus being on Taylor expansion, we rather propose to consider vector λ-terms as intermediate objects: the reduction relation induced on resource vectors by β-reduction through Taylor expansion should also contain β-reduction on vector terms — which is useful to understand what may go wrong. We still need to introduce some form of quotient in the syntax, though: otherwise, *e.g.*, $(\lambda x\, M + \lambda x\, N)\ P$ is normal.

We call *algebraic λ-terms* the elements of $\Lambda_{\mathbf{S}} := \Sigma_{\mathbf{S}}/{\simeq_+}$, where $\simeq_+$ denotes the least congruence containing the linearity equations of Fig. 2, group (b). We define the sets $\Sigma^{\mathrm{c}}_{\mathbf{S}}$ of *canonical terms* and $\Sigma^{\mathrm{s}}_{\mathbf{S}}$ of *simple canonical terms* simultaneously as follows:

$$\Sigma^{\mathrm{s}}_{\mathbf{S}} \ni S,T ::= x \mid \lambda x\, S \mid (S)\ M \qquad \text{and} \qquad \Sigma^{\mathrm{c}}_{\mathbf{S}} \ni M,N,P ::= S \mid 0 \mid a.M \mid M+N.$$

so that each algebraic term M admits a unique canonical $\simeq_+$-representative. Moreover, each simple canonical term $S \in \Sigma^{\mathrm{s}}_{\mathbf{S}}$ is of one of the following two forms:

- either $S = \lambda x_1 \cdots \lambda x_n\ (x)\ M_1 \cdots M_k$: S is a *head normal form*;
- or $S = \lambda x_1 \cdots \lambda x_n\ (\lambda x\, T)\ M_0 \cdots M_k$: $(\lambda x\, T)\ M_0$ is the *head redex* of S.

In the remaining of this paper we will systematically identify algebraic terms with their canonical representatives and keep $\simeq_+$ implicit. Moreover, we write $\Lambda^{\mathrm{s}}_{\mathbf{S}}$ for the set of simple algebraic λ-terms, *i.e.* those that admit a simple canonical representative.

By contrast with the above treatment of Taylor expansion, one can forget everything about coefficients and focus on sets of resource terms (identifying $\mathfrak{P}(\Delta)$ with $\mathbf{B}^\Delta$, we write, *e.g.*, $\lambda x\, \mathcal{S} = \{\lambda x\, s\ ;\ s \in \mathcal{S}\}$ if $\mathcal{S} \subseteq \Delta$):

▶ **Definition 19.** The Taylor support $\mathcal{T}(M) \subseteq \Delta$ of $M \in \Sigma_{\mathbf{S}}$ is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{T}(x) := \{x\} & \mathcal{T}(\lambda x\, M) := \lambda x\, \mathcal{T}(M) & \mathcal{T}((M)\ N) := \langle \mathcal{T}(M) \rangle\ \mathcal{T}(N)^! \\
\mathcal{T}(0) := \emptyset & \mathcal{T}(a.M) := \mathcal{T}(M) & \mathcal{T}(M+N) := \mathcal{T}(M) \cup \mathcal{T}(N).
\end{array}$$

$$\frac{s \Rightarrow_\partial \sigma' \quad \bar{t} \Rightarrow_\partial \bar{\tau}'}{\langle \lambda x\, s\rangle\ \bar{t} \Rightarrow_\partial \partial_x \sigma' \cdot \bar{\tau}'} \qquad \frac{}{x \Rightarrow_\partial x} \qquad \frac{s \Rightarrow_\partial \sigma'}{\lambda x\, s \Rightarrow_\partial \lambda x\, \sigma'} \qquad \frac{s \Rightarrow_\partial \sigma' \quad \bar{t} \Rightarrow_\partial \bar{\tau}'}{\langle s\rangle\ \bar{t} \Rightarrow_\partial \langle \sigma'\rangle\ \bar{\tau}'} \qquad \frac{s \Rightarrow_\partial \sigma' \quad \bar{t} \Rightarrow_\partial \bar{\tau}'}{[s]\cdot \bar{t} \Rightarrow_\partial [\sigma']\cdot \bar{\tau}'}$$

Figure 3 Rules of parallel resource reduction $\Rightarrow_\partial$.

Observe that $\mathcal{T}(M)$ is preserved under $\simeq_+$ so it is well defined on algebraic terms. Also observe that the inclusion $|\Theta(M)| \subseteq \mathcal{T}(M)$ might be strict. Moreover, the identity $\mathcal{T}(M[N/x]) = \mathcal{T}(M)[\mathcal{T}(N)/x]$ is established much more easily than Lemma 17.

We will make crucial use of the Taylor support operator in our study of Taylor expansion: although the latter is our subject of interest, the former is more robust where β-reduction is involved. For instance, an algebraic λ-term M is β-normal iff $\mathcal{T}(M)$ contains only normal resource terms: this fails for $|\Theta(M)|$ (and would also fail had we set $\mathcal{T}(0.N) = \emptyset$).

5 On the reduction of resource vectors

Observe that $\Theta((\lambda x\, M)\ N) = \langle \lambda x\, \Theta(M)\rangle\ \Theta(N)^!$ and $\Theta(M[N/x]) = \partial_x \Theta(M) \cdot \Theta(N)^!$. In order to simulate β-reduction through Taylor expansion we may consider the reduction given by $\epsilon \to \epsilon'$ iff $\epsilon = \sum_{i\in I} a_i.e_i$ and $\epsilon' = \sum_{i\in I} a_i.\epsilon'_i$ with $e_i \to_\partial^? \epsilon'_i$ for all $i \in I$. Observe indeed that, as soon as $(a_i.e_i)_{i\in I}$ is summable, the family $(a_i.\epsilon'_i)_{i\in I}$ is summable too, by Lemma 8. So we do not need any additional condition for this reduction step to be well defined.

This is not enough for simulating β-reduction, though, because we might need to reduce arbitrarily many redexes in parallel:

▶ **Example 20.** Observe that

$$\Theta((x)\ (\lambda y\, y)\ z) = \sum_{n,k_1,\dots,k_n \in \mathbf{N}} \frac{1}{n! k_1! \cdots k_n!} \cdot \langle x\rangle\ [\langle \lambda y\, y\rangle\ z^{k_1}, \dots, \langle \lambda y\, y\rangle\ z^{k_n}]$$

and $\Theta((x)\ z) = \sum_{n\in\mathbf{N}} \frac{1}{n!} \cdot \langle x\rangle\ z^n$. Intuitively, to simulate the β-reduction step $(x)\ (\lambda y\, y)\ z \to_\beta (x)\ z$, we proceed as follows: for each $n, k_1, \dots, k_n$,

- if $k_1 = \cdots = k_n = 1$, we reduce either each $\langle \lambda y\, y\rangle\ z^{k_j} = \langle \lambda y\, y\rangle\ [z]$ to z in parallel;
- otherwise $k_j \neq 1$ for some j, and we $\Rightarrow_\partial$-reduce $\langle \lambda y\, y\rangle\ z^{k_j}$ to 0.

▶ **Definition 21.** We define *parallel resource reduction* $\Rightarrow_\partial \subseteq (!)\Delta \times \mathbf{N}[(!)\Delta]$ by the rules of Fig. 3. We extend it to finite sums: $\sum_{i=1}^n e_i \Rightarrow_\partial \sum_{i=1}^n \epsilon'_i$ whenever $e_i \Rightarrow_\partial \epsilon'_i$ for $1 \le i \le n$.

Then $\to_\partial \subseteq \Rightarrow_\partial \subset \to_\partial^*$. Moreover, parallel resource reduction is confluent in a strong sense:

▶ **Definition 22.** We define the *full parallel reduct* $\mathsf{F}(e)$ of $e \in (!)\Delta$ inductively as follows:

$$\mathsf{F}(x) := x \qquad \mathsf{F}(\langle \lambda x\, s\rangle\ \bar{t}) := \partial_x \mathsf{F}(s) \cdot \mathsf{F}(\bar{t}) \qquad \mathsf{F}([s_1, \dots, s_n]) := [\mathsf{F}(s_1), \dots, \mathsf{F}(s_n)]$$
$$\mathsf{F}(\lambda x\, s) := \lambda x\, \mathsf{F}(s) \qquad \mathsf{F}(\langle u\rangle\ \bar{t}) := \langle \mathsf{F}(u)\rangle\ \mathsf{F}(\bar{t}) \quad \text{(if } u \text{ is not an abstraction)}$$

We extend F to sums, setting $\mathsf{F}(\sum_{i=1}^n e_i) = \sum_{i=1}^n \mathsf{F}(e_i)$. We obtain:

▶ **Lemma 23.** *For all $\epsilon, \epsilon' \in \mathbf{N}[(!)\Delta]$, if $\epsilon \Rightarrow_\partial \epsilon'$ then $\epsilon' \Rightarrow_\partial \mathsf{F}(\epsilon)$.*

Parallel reduction $\Rightarrow_\partial$ (like iterated reduction $\to_\partial^*$) lacks the combinatorial regularity properties of $\to_\partial$ given by Lemma 8: $e' \in (!)\Delta$ being fixed, there is no bound on the size of the $\Rightarrow_\partial$-antecedents of e', *i.e.* those $e \in (!)\Delta$ such that $e' \in |\epsilon'|$ with $e \Rightarrow_\partial \epsilon'$.

▶ **Example 24.** Fix $s \in \Delta$. Consider the sequences of resource terms given by $u_0(s) = v_0(s) = s$, $u_{n+1}(s) = \langle \lambda y\, y\rangle\ [u_n(s)]$ and $v_{n+1}(s) = \langle \lambda y\, v_n(s)\rangle\ []$. Then $u_n(s) \Rightarrow_\partial u_{n'}(s)$ and $v_n(s) \Rightarrow_\partial v_{n'}(s)$ for all $n \ge n' \in \mathbf{N}$. In particular $u_n(s) \Rightarrow_\partial s$ and $v_n(s) \Rightarrow_\partial s$.

It is thus no longer automatically possible to reduce all resource expressions in a resource vector simultaneously: consider, *e.g.*, $\sum_{n\in\mathbf{N}} u_n(x)$. Hence, in order to introduce a reduction relation on resource vectors by extending a reduction relation on resource expressions as above, we must in general impose the summability of the family of reducts as a side condition:

▶ **Definition 25.** Fix a relation $\to\, \subseteq (!)\Delta \times \mathbf{N}[(!)\Delta]$. For all $\epsilon, \epsilon' \in \mathbf{S}^{(!)\Delta}$, we write $\epsilon \widetilde{\to} \epsilon'$ whenever there exist a family $(a_i)_{i\in I} \in \mathbf{S}^I$, and summable families $(e_i)_{i\in I} \in (!)\Delta^I$ and $(\epsilon'_i)_{i\in I} \in \mathbf{N}[(!)\Delta]^I$ such that: $\epsilon = \sum_{i\in I} a_i.e_i$, $\epsilon' = \sum_{i\in I} a_i.\epsilon'_i$ and for all $i \in I$, $e_i \to^? \epsilon'_i$.

The necessity of such a side condition forbids confluence: *e.g.*, setting $\sigma = \sum_{n\in\mathbf{N}} u_n(v_n(x))$, we have $\sigma \widetilde{\Rightarrow_\partial} \sum_{n\in\mathbf{N}} u_n(x)$ and $\sigma \widetilde{\Rightarrow_\partial} \sum_{n\in\mathbf{N}} v_n(x)$, but since the only common reduct of $u_p(x)$ and $v_q(x)$ is x, there is no way to close this pair of reductions in general.

Notice that the families of terms we use in Example 24 involve chains of nested redexes, of unbounded length. The length of such chains in $e \in (!)\Delta$ is moreover bounded by $\mathbf{h}(e)$. Indeed, the size collapse induced by $\Rightarrow_\partial$ is essentially controlled by the height of expressions:

▶ **Lemma 26.** *If $e \Rightarrow_\partial \epsilon'$ and $e' \in |\epsilon'|$ then $\mathbf{s}(e) \leq 4^{\mathbf{h}(e)}\mathbf{s}(e')$.*

Proof. By induction on the reduction $e \Rightarrow_\partial \epsilon'$, using Lemma 6 in the redex case. ◀

Hence we can limit the size collapse if the reduction involves expressions of bounded height. We say $\epsilon \in \mathbf{S}^{(!)\Delta}$ is *bounded* if $\{\mathbf{h}(e) ; e \in |\epsilon|\}$ is finite. We then write $\mathbf{h}(\epsilon) = \max\{\mathbf{h}(e) ; e \in |\epsilon|\}$. Observe that $\Theta(M)$ is bounded for all $M \in \Lambda_\mathbf{S}$. For all $h \in \mathbf{N}$, we write $(!)\Delta_h := \{e \in (!)\Delta ; \mathbf{h}(e) \leq h\}$ and then $(!)\mathfrak{B} := \bigcup_{h\in\mathbf{N}} \mathfrak{P}((!)\Delta_h)$ is a finiteness structure: $(!)\mathfrak{B} = \{(!)\Delta_h ; h \in \mathbf{N}\}^{\perp\perp}$. The semimodule of bounded resource vectors is then $\mathbf{S}\langle(!)\mathfrak{B}\rangle$. We can moreover show that reducing expressions of bounded height generates expressions of bounded height (with a greater bound):

▶ **Lemma 27.** *If $e \Rightarrow_\partial \epsilon'$ and $e' \in |\epsilon'|$ then $\mathbf{h}(e') \leq 2^{\mathbf{h}(e)}\mathbf{h}(e)$.*

▶ **Corollary 28.** *If $(e_i)_{i\in I} \in (!)\Delta_h^I$ is summable and $e_i \Rightarrow_\partial \epsilon'_i$ for all $i \in I$, then $(\epsilon'_i)_{i\in I}$ is summable and $\epsilon'_i \in \mathbf{N}[(!)\Delta_{2^h h}]$ for all $i \in I$.*

In particular, for a fixed $h \in \mathbf{N}$, $(\mathsf{F}(e))_{e\in(!)\Delta_h}$ is summable. Hence F extends to a linear-continuous map $\mathsf{F} : \mathbf{S}^{(!)\Delta_h} \to \mathbf{S}^{(!)\Delta_{2^h h}}$, and more generally to a map $\mathsf{F} : \mathbf{S}\langle(!)\mathfrak{B}\rangle \to \mathbf{S}\langle(!)\mathfrak{B}\rangle$.

Bounded vectors, however, may not be preserved under $\widetilde{\Rightarrow_\partial}$. Indeed, like in the algebraic λ-calculus, reduction can interact with the semimodule structure of $\mathbf{S}^{(!)\Delta}$: we can reproduce Example 18 in $\mathbf{S}^{(!)\Delta}$ through Taylor expansion (see Corollary 34). More directly, we can use the terms of Example 24: for any $s \in \Delta$, setting $\sigma = \sum_{n\in\mathbf{N}} u_{n+1}(s) \in \mathbf{Z}^\Delta$ we obtain $0 = \sigma - \sigma \widetilde{\Rightarrow_\partial} \sum_{n\in\mathbf{N}} u_n(s) - \sigma = s$. Of course, this kind of issue does not arise when the semiring of coefficients is *zerosumfree*, *i.e.*, for all $a, b \in \mathbf{S}$, $a + b = 0$ iff $a = b = 0$:

▶ **Lemma 29.** *Assume $\mathbf{S}$ is zerosumfree and fix a relation $\to\, \subseteq (!)\Delta \times \mathbf{N}[(!)\Delta]$. If $\epsilon \widetilde{\to} \epsilon'$ then, for all $e' \in |\epsilon'|$ there exists $e \in |\epsilon|$ and $\epsilon_0 \in \mathbf{N}[(!)\Delta]$ such that $e \to^? \epsilon_0$ and $e' \in |\epsilon_0|$.*

Various approaches to get rid of this restriction in the setting of the algebraic λ-calculus [24, 2, 1, 7] may be adapted to the reduction of resource vectors. The linear-continuity of the resource λ-calculus allows us to propose a novel approach: consider possible restrictions on the families of resource expressions simultaneously reduced in a $\widetilde{\to}$-step.

▶ **Definition 30.** Fix a relation $\to\, \subseteq (!)\Delta \times \mathbf{N}[(!)\Delta]$. If $\mathcal{E} \subseteq (!)\Delta$, we write $e \to_{|\mathcal{E}} \epsilon'$ iff $e \to \epsilon'$ and $e \in \mathcal{E}$, and then $\widetilde{\to}_\mathcal{E} := \widetilde{\to_{|\mathcal{E}}}$. If $\mathfrak{E} \subseteq \mathfrak{P}((!)\Delta)$, we then write $\widetilde{\to}_\mathfrak{E}$ for $\bigcup_{\mathcal{E}\in\mathfrak{E}} \widetilde{\to}_\mathcal{E}$.

$$\frac{S \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{(\lambda x\, S)\ N \Rightarrow_\beta M'[N'/x]} \qquad \frac{}{x \Rightarrow_\beta x} \qquad \frac{S \Rightarrow_\beta M'}{\lambda x\, S \Rightarrow_\beta \lambda x\, M'} \qquad \frac{S \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{(S)\ N \Rightarrow_\beta (M')\ N'}$$

$$\frac{}{0 \Rightarrow_\beta 0} \qquad \frac{M \Rightarrow_\beta M'}{a.M \Rightarrow_\beta a.M'} \qquad \frac{M \Rightarrow_\beta M' \quad N \Rightarrow_\beta N'}{M + N \Rightarrow_\beta M' + N'}$$

Figure 4 Rules of parallel β-reduction $\Rightarrow_\beta$ of algebraic λ-terms.

In particular, we have $\widetilde{\to}_{(!)\Delta} = \widetilde{\to}$ and $\widetilde{\to}_{\mathcal{E}} \subseteq \widetilde{\to} \cap (\mathbf{S}^{\mathcal{E}} \times \mathbf{S}^{(!)\Delta})$, but in general the reverse inclusion holds only if $\mathbf{S}$ is zerosumfree: in this latter case $\epsilon \mathrel{\widetilde{\Rightarrow}_\partial} \epsilon'$ iff $\epsilon \mathrel{\widetilde{\Rightarrow}_{\partial|\epsilon|}} \epsilon'$.

We call *reduction structure* any set $\mathfrak{E} \subseteq \mathfrak{P}((!)\Delta)$ such that: (i) $\mathfrak{P}_f((!)\Delta) \subseteq \mathfrak{E}$; (ii) $\mathfrak{E}$ is closed under finite unions; (iii) $\mathfrak{E}$ is downwards closed for inclusion; (iv) for all $\mathcal{E} \in \mathfrak{E}$, $\{e' \in (!)\Delta\ ;\ e' \in |\epsilon'| \text{ with } e \in \mathcal{E} \text{ and } e \to \epsilon'\} \in \mathfrak{E}$. Observe that the first three requirements are automatically satisfied by finiteness structures.

It follows from Lemma 27 that $(!)\mathfrak{B}$ is a $\Rightarrow_\partial$-reduction structure. Let us call *bounded reduction structure* any $\Rightarrow_\partial$-reduction structure $\mathfrak{E}$ such that $\mathfrak{E} \subseteq (!)\mathfrak{B}$. Lemmas 23 and 26 then entail the strong confluence of $\widetilde{\Rightarrow}_{\partial\mathfrak{E}}$:

▶ **Theorem 31.** *For any bounded reduction structure $\mathfrak{E}$, if $\epsilon \mathrel{\widetilde{\Rightarrow}_{\partial\mathfrak{E}}} \epsilon'$ then $\epsilon' \mathrel{\widetilde{\Rightarrow}_{\partial\mathfrak{E}}} \mathsf{F}(\epsilon)$.*

6 Taylor expansion and β-reduction

From now on, for all $M, N \in \Lambda_{\mathbf{S}}$, we write $M \mathrel{\widetilde{\Rightarrow}_\partial} N$ if $\Theta(M) \mathrel{\widetilde{\Rightarrow}_\partial} \Theta(N)$. More generally, for all $M \in \Lambda_{\mathbf{S}}$ and all $\sigma \in \mathbf{S}^{(!)\Delta}$, we write $M \mathrel{\widetilde{\Rightarrow}_\partial} \sigma$ (resp. $\sigma \mathrel{\widetilde{\Rightarrow}_\partial} M$) if $\Theta(M) \mathrel{\widetilde{\Rightarrow}_\partial} \sigma$ (resp. $\sigma \mathrel{\widetilde{\Rightarrow}_\partial} \Theta(M)$). We show that $M \mathrel{\widetilde{\Rightarrow}_\partial} N$ as soon as $M \Rightarrow_\beta N$ where $\Rightarrow_\beta \subseteq \Lambda_{\mathbf{S}} \times \Lambda_{\mathbf{S}}$ is the parallel β-reduction inductively defined on algebraic λ-terms by the rules of Fig. 4.

▶ **Lemma 32.** *If $\sigma \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{S}}} \sigma'$ and $\overline{\tau} \mathrel{\widetilde{\Rightarrow}_{\partial\overline{\mathcal{T}}}} \overline{\tau}'$ then we have:*

$$\langle\lambda x\, \sigma\rangle\ \overline{\tau} \mathrel{\widetilde{\Rightarrow}_{\partial\langle\lambda x\,\mathcal{S}\rangle\,\overline{\mathcal{T}}}} \partial_x\sigma' \cdot \overline{\tau}' \qquad \lambda x\, \sigma \mathrel{\widetilde{\Rightarrow}_{\partial\lambda x\,\mathcal{S}}} \lambda x\, \sigma' \qquad \langle\sigma\rangle\ \overline{\tau} \mathrel{\widetilde{\Rightarrow}_{\partial\langle\mathcal{S}\rangle\,\overline{\mathcal{T}}}} \langle\sigma'\rangle\ \overline{\tau}' \qquad \sigma^! \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{S}^!}} \sigma'^!.$$

Moreover, if $\epsilon \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{E}}} \epsilon'$ and $\phi \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{F}}} \phi'$ then $a.\epsilon \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{E}}} a.\epsilon'$ and $\epsilon + \phi \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{E}\cup\mathcal{F}}} \epsilon' + \phi'$.

Proof. By the multilinear-continuity of syntactic constructs (for $\sigma^! \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{S}^!}} \sigma'^!$ we first prove $\sigma^n \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{S}^!}} \sigma'^n$ for each $n \in \mathbf{N}$) and the fact that summable families form an $\mathbf{S}$-semimodule. ◀

▶ **Theorem 33.** *If $M \Rightarrow_\beta M'$ then $M \mathrel{\widetilde{\Rightarrow}_{\partial\mathcal{T}(M)}} M'$.*

Proof. By induction on the reduction $M \Rightarrow_\beta M'$ using Lemma 32 in each case. ◀

▶ **Corollary 34.** *If $M \Rightarrow_\beta M'$ then $M \mathrel{\widetilde{\Rightarrow}_{\partial(!)\mathfrak{B}}} M'$.*

In particular, if $1 \in \mathbf{S}$ admits an opposite element $-1 \in \mathbf{S}$ then $\widetilde{\Rightarrow}_{\partial(!)\mathfrak{B}}$ is degenerate. Indeed, we can consider $\Rightarrow_\beta$ up to the equality of vector λ-terms by setting $M \Rightarrow_{\widetilde{\beta}} N$ if there are $M' \simeq_{\mathrm{v}} M$ and $N' \simeq_{\mathrm{v}} N$ such that $M' \Rightarrow_\beta N'$. Since $\simeq_\Theta$ subsumes $\simeq_{\mathrm{v}}$, we have $M \mathrel{\widetilde{\Rightarrow}_{\partial(!)\mathfrak{B}}} N$ as soon as $M \Rightarrow_{\widetilde{\beta}} N$. If $-1 \in \mathbf{S}$, we have $M \Rightarrow_{\widetilde{\beta}}^* N$ for all $M, N \in \Lambda_{\mathbf{S}}$ by Example 18, hence $M \mathrel{\widetilde{\Rightarrow}^*_{\partial(!)\mathfrak{B}}} N$.

Even assuming $\mathbf{S}$ is zerosumfree, Taylor expansions are not stable under $\widetilde{\Rightarrow}_\partial$: if $M \mathrel{\widetilde{\Rightarrow}_{\partial\mathfrak{B}}} \sigma'$ there is no reason why σ' would be the Taylor expansion of a term. We do know, however, that $\sigma' \mathrel{\widetilde{\Rightarrow}_{\partial\mathfrak{B}}} \mathsf{F}(\Theta(M))$ and $M \mathrel{\widetilde{\Rightarrow}_{\partial\mathfrak{B}}} \mathsf{F}(\Theta(M))$: we show the latter is the image of a $\Rightarrow_\beta$-step.

▶ **Definition 35.** We define the *full parallel reduct* $\mathsf{F}(S)$ of a simple term S, resp. $\mathsf{F}(M)$ of an algebraic term M, by mutual induction. On algebraic terms, we set $\mathsf{F}(0) := 0$, $\mathsf{F}(a.M) := a.\mathsf{F}(M)$, $\mathsf{F}(M+N) := \mathsf{F}(M)+\mathsf{F}(N)$, On simple terms:

$$\mathsf{F}(x) := x \qquad \mathsf{F}((\lambda x\, S)\ N) := \mathsf{F}(S)[\mathsf{F}(N)/x]$$
$$\mathsf{F}(\lambda x\, S) := \lambda x\, \mathsf{F}(S) \qquad \mathsf{F}((T)\ N) := (\mathsf{F}(T))\ \mathsf{F}(N) \quad \text{(if } T \text{ is not an abstraction).}$$

▶ **Lemma 36.** *If $M \Rightarrow_\beta M'$ then $M' \Rightarrow_\beta \mathsf{F}(M)$. Moreover, for all $M \in \Lambda_{\mathbf{S}}$, $\mathsf{F}(\Theta(M)) = \Theta(\mathsf{F}(M))$.*

Proof. The reduction $M' \Rightarrow_\beta \mathsf{F}(M)$ is constructed by induction on $M \Rightarrow_\beta M'$. The identity $\mathsf{F}(\Theta(M)) = \Theta(\mathsf{F}(M))$ is proved by induction on M: each case is similar to Lemma 32. ◀

▶ **Lemma 37.** *For all bounded reduction structure $\mathfrak{S}$, if $M \,\widetilde{\Rightarrow}_{\partial\mathfrak{S}}\, \sigma'$ then $\sigma' \,\widetilde{\Rightarrow}_{\partial\mathfrak{S}}\, \mathsf{F}(M)$.*

Proof. By Theorem 31, $\sigma' \,\widetilde{\Rightarrow}_{\partial\mathfrak{S}}\, \mathsf{F}(\Theta(M))$ and we conclude by the previous lemma. ◀

▶ **Corollary 38.** *For all bounded reduction structure $\mathfrak{S}$, if $M \,\widetilde{\Rightarrow}^n_{\partial\mathfrak{S}}\, \sigma'$ then $\sigma' \,\widetilde{\Rightarrow}^n_{\partial\mathfrak{S}}\, \mathsf{F}^n(M)$.*

We write $\simeq_\beta$ for the symmetric, reflexive and transitive closure of $\Rightarrow_\beta$. Similarly, if $\mathfrak{E}$ is a bounded reduction structure, we write $\simeq_{\partial\mathfrak{E}}$ for the induced equivalence on $\mathbf{S}\langle\mathfrak{E}\rangle$: it follows from Theorem 31 and Corollary 38 that $\epsilon \simeq_{\partial\mathfrak{E}} \epsilon'$ iff $\epsilon' \,\widetilde{\Rightarrow}^*_{\partial\mathfrak{E}}\, \mathsf{F}^n(\epsilon)$ for some $n \in \mathbf{N}$. If $M \in \Lambda_{\mathbf{S}}$ is normalizable, we write $\mathsf{NF}(M)$ for its normal form and obtain that, for all N such that $M \simeq_{\partial\mathfrak{B}} N$, we have $\mathsf{NF}(M) \,\widetilde{\Rightarrow}_{\partial\mathfrak{B}}\, \mathsf{F}^n(N)$ for some $n \in \mathbf{N}$. In particular, if $\mathbf{S}$ is zerosumfree, $\mathsf{NF}(M) \simeq_\Theta \mathsf{F}^n(N)$. If moreover $M, N \in \Lambda$, we obtain $M \simeq_\beta N$. The next section will allow us to establish a similar conservativity result, without any assumption on $\mathbf{S}$, at the cost of restricting the reduction relation to normalizable resource vectors.

7 Normalization and Taylor expansion commute

Let us first remark that the function $e \in (!)\Delta \mapsto \mathsf{NF}(e) \in \mathbf{N}[(!)\Delta]$ is not summable.

▶ **Example 39.** Recall the family $u_n(s)$ of terms from Example 24: if s is normal, we have $\mathsf{NF}(u_n(x)) = s$ for all $n \in \mathbf{N}$ and the family $(s)_{n\in\mathbf{N}}$ is obviously not summable. With a bit more work, one can show that there are infinitely many $t \in |\Theta(\infty_x)|$ such that $\mathsf{NF}(t) = x$, so NF is not summable even when restricted to the image of Taylor expansion.

We say $\epsilon \in \mathbf{S}^{(!)\Delta}$ is *normalizable* whenever the family $(\mathsf{NF}(e))_{e\in|\epsilon|}$ is summable. In this case, we write $\mathsf{NF}(\epsilon) := \sum_{e\in(!)\Delta} \epsilon_e.\mathsf{NF}(e)$.

Recall from Section 3 that $e \geq_\partial e'$ iff $e \to^*_\partial \epsilon'$ with $e' \in |\epsilon'|$. If $e \in (!)\Delta$, we write $\uparrow e := \{e' \in (!)\Delta\ ;\ e' \geq_\partial e\}$. Then ϵ is normalizable iff for each normal resource expression e, $|\epsilon| \cap \uparrow e$ is finite: writing $(!)\mathcal{N} = \{e \in (!)\Delta\ ;\ e \text{ is normal}\}$ and $(!)\mathfrak{N} = \{\uparrow e\ ;\ e \in (!)\mathcal{N}\}^\perp$, we obtain that $\mathbf{S}\langle(!)\mathfrak{N}\rangle$ is the finiteness space of normalizable resource vectors (in other words, ϵ is normalizable iff $|\epsilon| \in (!)\mathfrak{N}$). For our study of hereditarily determinable terms in Section 8, it will be useful to decompose $(!)\mathfrak{N}$ into a decreasing sequence of finiteness structures.

▶ **Definition 40.** We define the applicative depth $\mathbf{d}(e) \in \mathbf{N}$ of a resource expression $e \in (!)\Delta$ inductively as follows (assuming $\bar{t} = [t_1, \ldots, t_n]$):

$$\mathbf{d}(x) = 1 \quad \mathbf{d}(\lambda x\, s) = \mathbf{d}(s) \quad \mathbf{d}(\langle s\rangle\ \bar{t}) = \max(\mathbf{d}(s), \mathbf{d}(\bar{t})+1) \quad \mathbf{d}(\bar{t}) = \max(\mathbf{d}(t_1), \ldots, \mathbf{d}(t_n)).$$

We write $(!)\mathcal{N}_d = \{e \in (!)\mathcal{N}\ ;\ \mathbf{d}(e) \leq d\}$: $(!)\mathcal{N}_d$ is the set of normal resource expressions of applicative depth at most d, so that $(!)\mathcal{N} = \bigcup_{d\in\mathbf{N}}(!)\mathcal{N}_d$. We then write $(!)\mathfrak{N}_d = \{\uparrow e\ ;\ e \in (!)\mathcal{N}_d\}^\perp$ so that $(!)\mathfrak{N} = \bigcap_{d\in\mathbf{N}}(!)\mathfrak{N}_d$. Each finiteness structure $(!)\mathfrak{N}_d$ is moreover a reduction structure for any reduction relation contained in $\to_\partial^*$ (and so is $(!)\mathfrak{N}$):

▶ **Lemma 41.** *If $\mathcal{E} \in (!)\mathfrak{N}_d$ then $\downarrow\mathcal{E} := \{e' \in (!)\Delta\ ;\ e \geq_\partial e'$ with $e \in \mathcal{E}\} \in (!)\mathfrak{N}_d$.*

Proof. Let $e'' \in (!)\mathcal{N}_d$ and $e' \in \downarrow\mathcal{E} \cap \uparrow e''$. Necessarily, there is $e \in \mathcal{E}$ such that $e \geq_\partial e'$. Then $e \in \mathcal{E} \cap \uparrow e''$: since $\mathcal{E} \in (!)\mathfrak{N}_d$, there are finitely many possible values for e hence for e'. ◀

▶ **Lemma 42.** *If $\epsilon \in \mathbf{S}\langle(!)\mathfrak{N}\rangle$ and $\epsilon \mathrel{\widetilde{\Rrightarrow}_{\partial(!)\mathfrak{N}}} \epsilon'$ then $\epsilon' \in \mathbf{S}\langle(!)\mathfrak{N}\rangle$ and $\mathsf{NF}(\epsilon) = \mathsf{NF}(\epsilon')$.*

Proof. Assume there exists $\mathcal{E} \in (!)\mathfrak{N}$, a family $\overrightarrow{a} = (a_i)_{i\in I} \in \mathbf{S}^I$, and summable families $(e_i)_{i\in I} \in (!)\Delta^I$ and $(\epsilon'_i)_{i\in I} \in \mathbf{N}[(!)\Delta]^I$ such that: $\epsilon = \sum_{i\in I} a_i.e_i$, $\epsilon' = \sum_{i\in I} a_i.\epsilon'_i$ and for all $i \in I$, $e_i \in \mathcal{E}$ and $e_i \mathrel{\widetilde{\Rrightarrow}_\partial} \epsilon'_i$. We obtain that $\mathcal{E}' := \bigcup_{i\in I} |\epsilon'_i| \in (!)\mathfrak{N}$ by Lemma 41, hence $\epsilon' \in \mathbf{S}\langle(!)\mathfrak{N}\rangle$ since $|\epsilon'| \subseteq \mathcal{E}'$. Then, by the linear-continuity of NF on $\mathbf{S}^{\mathcal{E}'}$, $\mathsf{NF}(\epsilon) = \sum_{i\in I} a_i.\mathsf{NF}(e_i) = \sum_{i\in I} a_i.\mathsf{NF}(\epsilon'_i) = \mathsf{NF}(\sum_{i\in I} a_i.\epsilon'_i) = \mathsf{NF}(\epsilon')$. ◀

The following three lemmas are the key steps of a reducibility argument: like in Ehrhard's work for the typed case [10], or our previous work for the strongly normalizable case [22], each $(!)\mathfrak{N}_d$ is the analogue of a reducibility candidate. Similar results for $(!)\mathfrak{N}$ are immediately derived from those. In our setting, the proof of Lemma 45 is made easier by the fact that, for $\mathcal{E} \in (!)\mathfrak{N}_d$, we require $\mathcal{E} \cap \uparrow e$ to be finite only when e is normal.

▶ **Lemma 43.** *If $\mathcal{S} \in \mathfrak{N}_d$ then $\lambda x\, \mathcal{S} \in \mathfrak{N}_d$.*

▶ **Lemma 44.** *If $\mathcal{T}_1, \ldots, \mathcal{T}_n \in \mathfrak{N}_d$ then $\langle x\rangle\ \mathcal{T}_1^! \cdots \mathcal{T}_n^! \in \mathfrak{N}_{d+1}$.*

▶ **Lemma 45.** *If $\left\langle \partial_x \mathcal{S} \cdot \overline{\mathcal{T}}_0 \right\rangle\ \overline{\mathcal{T}}_1 \cdots \overline{\mathcal{T}}_n \in \mathfrak{N}_d$ then $\langle \lambda x\, \mathcal{S}\rangle\ \overline{\mathcal{T}}_0\, \overline{\mathcal{T}}_1 \cdots \overline{\mathcal{T}}_n \in \mathfrak{N}_d$*

Observe that the weak normalizability of algebraic λ-terms is not stable under $\simeq_{\mathrm{v}}$ because of the identity $0 \simeq_{\mathrm{v}} 0.M$. Since we work up to $\simeq_+$ only, a general standardization argument [21] applies and we have that $M \in \Lambda_{\mathbf{S}}$ is normalizable iff the left reduction strategy terminates, *i.e.* there exists $k \in \mathbf{N}$ such that $\mathsf{L}^k(M)$ is normal, where L is defined as follows:

▶ **Definition 46.** We define the *left reduct* $\mathsf{L}(S)$ of a simple term S, resp. $\mathsf{L}(M)$ of an algebraic term M, by mutual induction. On algebraic terms, we set $\mathsf{L}(0) := 0$, $\mathsf{L}(a.M) := a.\mathsf{L}(M)$, $\mathsf{L}(M + N) := \mathsf{L}(M) + \mathsf{L}(N)$, On simple terms:

$$\mathsf{L}(\lambda x\, S) := \lambda x\, \mathsf{L}(S) \qquad \mathsf{L}((x)\ M_1 \cdots M_n) := (x)\ \mathsf{L}(M_1) \cdots \mathsf{L}(M_n)$$
$$\mathsf{L}((\lambda x\, S)\ M_0\, M_1 \cdots M_n) := (S[M_0/x])\ M_1 \cdots M_n$$

▶ **Theorem 47.** *If M is normalizable, then $\mathcal{T}(M) \in \mathfrak{N}$, and $\Theta(M) \in \mathbf{S}\langle\mathfrak{N}\rangle$.*

Proof. By induction on k such that $\mathsf{L}^k(M)$ is normal, then on M. We use Lemmas 43 to 45 for $\mathfrak{N}$ in the case of simple terms, and the fact that $\mathfrak{N}$ is a finiteness structure otherwise. ◀

Writing $\mathsf{NF}(M)$ for the normal form of M, it remains to prove that $\Theta(\mathsf{NF}(M)) = \mathsf{NF}(\Theta(M))$.

▶ **Lemma 48.** *We define the left reduct of a resource expression inductively as follows:*

$$\mathsf{L}(\lambda x\, s) := \lambda x\, \mathsf{L}(s) \qquad \mathsf{L}(\langle x\rangle\ \bar{t}_1 \cdots \bar{t}_n) := \langle x\rangle\ \mathsf{L}(\bar{t}_1) \cdots \mathsf{L}(\bar{t}_n)$$
$$\mathsf{L}([t_1, \ldots, t_n]) := [\mathsf{L}(t_1), \ldots, \mathsf{L}(t_n)] \qquad \mathsf{L}(\langle \lambda x\, s\rangle\ \bar{t}_0\, \bar{t}_1 \cdots \bar{t}_n) := \left\langle \partial_x s \cdot \bar{t}_0 \right\rangle\ \bar{t}_1 \cdots \bar{t}_n$$

Then for all resource expression $e \in (!)\Delta$, $e \Rightarrow_\partial \mathsf{L}(e)$. In particular $\mathsf{NF}(e) = \mathsf{NF}(\mathsf{L}(e))$. If moreover $e' \in |\mathsf{L}(e)|$ then $\mathbf{s}(e) \leq 4\mathbf{s}(e')$.

$$\frac{}{M \Downarrow_0} \quad \frac{M \Uparrow}{M \Downarrow_d} \quad \frac{S \Downarrow_d}{\lambda x\, S \Downarrow_d} \quad \frac{M_1 \Downarrow_d \quad \cdots \quad M_n \Downarrow_d}{(x)\ M_1 \cdots M_n \Downarrow_{d+1}} \quad \frac{M \Downarrow_d}{a.M \Downarrow_d} \quad \frac{M \Downarrow_d \quad N \Downarrow_d}{M + N \Downarrow_d} \quad \frac{(S[M_0/x])\ M_1 \cdots M_n \Downarrow_d}{(\lambda x\, S)\ M_0 \cdots M_n \Downarrow_d}$$

Figure 5 Rules for determinable terms

Proof. Similar to Lemmas 23 and 26. ◀

As a consequence $(\mathsf{L}(e))_{e\in(!)\Delta}$ is summable, so that $\mathsf{L}(\Theta(M)) = \Theta(\mathsf{L}(M))$, for any $M \in \Lambda_{\mathbf{S}}$.

▶ **Theorem 49.** *If M is normalizable, then $\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{NF}(M))$.*

Proof. Assume $\mathsf{NF}(M) = \mathsf{L}^k(M)$. By Theorem 47 and Lemma 42, $\mathsf{L}^k(\Theta(M)) \in \mathbf{S}\langle\mathfrak{N}\rangle$ and $\mathsf{NF}(\Theta(M)) = \mathsf{NF}(\mathsf{L}^k(\Theta(M)))$. Then $\Theta(\mathsf{NF}(M)) = \Theta(\mathsf{L}^k(M)) = \mathsf{L}^k(\Theta(M)) = \mathsf{NF}(\Theta(M))$. ◀

If M and N are normalizable and $M \simeq_{\partial\mathfrak{N}\cap\mathfrak{B}} N$, Theorems 31 and 33, and Corollary 38, entail that $\mathsf{L}^k(\mathsf{NF}(M)) \simeq_{\partial\mathfrak{N}\cap\mathfrak{B}} \mathsf{NF}(N)$ for some $k \in \mathbf{N}$. Since $\mathsf{L}^k(\mathsf{NF}(M)) = \mathsf{NF}(M)$, the previous Theorem entails $\mathsf{NF}(M) \simeq_\Theta \mathsf{NF}(N)$. In particular if M and N are pure λ-terms, we have $M \simeq_\beta N$. It follows that $\simeq_{\partial\mathfrak{N}\cap\mathfrak{B}}$ is a non trivial extension of $\simeq_\beta$ on normalizable algebraic λ-terms. We can extend this conservativity result to non-normalizing pure λ-terms thanks to previous work by Ehrhard and Regnier:

▶ **Theorem 50** ([13, 12])**.** *For all pure λ-term $M \in \Lambda$, $\mathcal{T}(M) \in \mathfrak{N}$ and $\mathsf{NF}(\Theta(M)) = \Theta(\mathsf{BT}(M))$ where $\mathsf{BT}(M)$ denotes the Böhm tree of M.*

▶ **Lemma 51.** *If $M, N \in \Lambda$ and $M \simeq_{\partial\mathfrak{N}\cap\mathfrak{B}} N$ then $\mathsf{BT}(M) = \mathsf{BT}(N)$.*

Proof. By confluence, there is σ such that $M \mathrel{\widetilde{\Rightarrow}}^*_{\partial\mathfrak{N}\cap\mathfrak{B}} \sigma$ and $N \mathrel{\widetilde{\Rightarrow}}^*_{\partial\mathfrak{N}\cap\mathfrak{B}} \sigma$, and by Theorems 49 and 50 we obtain: $\Theta(\mathsf{BT}(M)) = \mathsf{NF}(\Theta(M)) = \mathsf{NF}(\sigma) = \mathsf{NF}(\Theta(N)) = \Theta(\mathsf{BT}(N))$. We conclude since Θ is injective on Böhm trees. ◀

8 Normal form of Taylor expansion, *façon* Böhm trees

In the ordinary λ-calculus, head normalizable terms are exactly those with a non trivial Böhm tree. This is reflected via Taylor expansion: $\mathsf{NF}(\Theta(M)) = 0$ iff M has no head normal form. In the non-uniform case, we take this characterization as a definition: we say an algebraic λ-term $M \in \Lambda_{\mathbf{S}}$ is *Taylor-unsolvable* and write $M \Uparrow$ if $\mathsf{NF}(s) = 0$ for all $s \in \mathcal{T}(M)$.

▶ **Definition 52.** Let $M \in \Lambda_{\mathbf{S}}$ be an algebraic λ-term. We say M is *d-determinable* if the judgement $M \Downarrow_d$ can be derived inductively from the rules of Fig. 5. We say M is *hereditarily determinable* and write $M \Downarrow_\omega$ if $M \Downarrow_d$ for all $d \in \mathbf{N}$. We say M is in *d-determinate form* and write $M\,\mathsf{df}_d$ if $M \Downarrow_d$ is derivable from the above rules excluding the last one (head redex).

We have $M \Downarrow_\omega$ whenever M is normalizable. Moreover, $M \Downarrow_\omega$ for all $M \in \Lambda$.

▶ **Lemma 53.** *If $M \Downarrow_d$ then $\mathcal{T}(M) \in \mathfrak{N}_d$.*

Proof. By induction on the derivation of $M \Downarrow_d$, using in particular Lemmas 43 to 45. ◀

▶ **Lemma 54.** *If $M\,\mathsf{df}_d$, $s \in \mathcal{T}(M)$ and $s \geq_\partial s'$ with $s' \in \mathcal{N}_d$, then $s = s'$. If moreover $M \Rightarrow_\beta M'$ then $\Theta(M)_{s'} = \Theta(M')_{s'}$.*

Proof. By induction on the reduction $s \to^*_\partial \sigma'$ with $s' \in |\sigma'|$. ◀

▶ **Theorem 55.** *If $M \Downarrow_\omega$ then $\mathcal{T}(M) \in \mathfrak{N}$. Moreover, if $s \in \mathcal{N}_d$ then for all M' such that $M \simeq_\beta M'$ and $M'\,\mathsf{df}_d$ we have $\mathsf{NF}(\Theta(M))_s = \Theta(M')_s$.*

Proof. The fact that $\mathcal{T}(M) \in \mathfrak{N}$ follows from Lemma 53. It is easy to prove by induction on d that there exists $k \in \mathbf{N}$ such that $\mathsf{L}^k(M)\,\mathsf{df}_d$. Let $s \in \mathcal{N}_d$: we first prove $\mathsf{NF}(\Theta(M))_s = \Theta(\mathsf{L}^k(M))_s$. Since $M \Rightarrow_\beta^* \mathsf{L}^k(M)$, Theorem 33 and Lemma 42 entail $\mathsf{NF}(\Theta(M)) = \mathsf{NF}(\Theta(\mathsf{L}^k(M)))$. We then obtain $\mathsf{NF}(\Theta(\mathsf{L}^k(M)))_s = \Theta(\mathsf{L}^k(M))_s$, using Lemma 54. Now assume $M \simeq_\beta M'$ and $M'\,\mathsf{df}_d$. By the confluence of $\Rightarrow_\beta$, there exists $M'' \in \Lambda_{\mathbf{S}}$ such that $\mathsf{L}^k(M) \Rightarrow_\beta^* M''$ and $M' \Rightarrow_\beta^* M''$: we conclude by Lemma 54 again. ◀

The d-determinate forms of an hereditarily determinable term M differ only by subterms that are Taylor-unsolvable or at applicative depth greater than d. If we cut out unsolvable subterms, and replace the arguments of applications at depth d by 0, the common value is a normal term M_d with $\mathbf{d}(M_d) \leq d$: this is exactly the analogue of a Böhm tree approximant. In particular, Theorem 55 is a proper generalization of Theorem 50, and hereditarily determinable terms form the maximal class for which such an extension may hold: without any topological information on scalars, nothing can be said about infinite sums of head normal forms.

9 Conclusion

We have presented a notion of reduction on resource vectors, simulating β-reduction, and proved that Taylor expansion and normalization commute in a generic non-uniform setting. In the final section we have outlined the basic blocks of a generalization of Böhm trees. This construction is essentially *ad-hoc*: its only purpose is to generalize Theorem 50. Our main achievement here is conceptual: we have formalized the idea that *the normal form of Taylor expansion is an appropriate notion of denotation in a non-uniform setting.*

A natural question to ask is how normalizing the Taylor expansion compares with pre-existing notions of denotation in various non-deterministic settings: must-non-deterministic Böhm trees [6], probabilistic Böhm trees [21], weighted relational models [4, 20, 19], *etc.* Even closer to our work, Tsukada, Asada and Ong have recently established [23] the commutation between computing Böhm trees of non-deterministic λ-terms and Taylor expansion with coefficients taken in the complete semiring of positive reals, where all sums converge.

Their approach is based on an precise description of the relationship between the coefficients of resource terms in the expansion of a term and those in the expansion of its Böhm-tree, using a *rigid* Taylor expansion as an intermediate step: this avoids the ambiguity between the sums of coefficients generated by redundancies in the expansion and those representing non-deterministic superpositions. Their work can thus be considered as a refinement of Ehrhard and Regnier's method, that they are moreover able to generalize to the non-deterministic case provided the semiring of scalars is complete. By contrast, our approach is focused on β-reduction and identifies a class of algebraic λ-terms for which the normalization of Taylor expansion converges independently from the topology on scalars. It seems only natural to study the connections between both approaches, in particular to tackle the case of weighted non-determinism in a complete semiring, as a first step towards the treatment of probabilistic or quantum superposition.

We are moreover investigating the extension of our approach to linear logic proof nets, as well as to infinitary λ-calculus [18]: the Taylor expansion operator extends naturally to both proof nets and infinitary λ-terms, and preliminary work suggests that our study of reduction under Taylor expansion is robust enough that it could be adapted to both settings.

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References

1 M. Alberti. *On operational properties of quantitative extensions of λ-calculus.* PhD thesis, Aix Marseille Université and Università di Bologna, 2014.
2 P. Arrighi and G. Dowek. Linear-algebraic lambda-calculus: higher-order, encodings, and confluence. In *RTA 2008*, volume 5117 of *LNCS*. Springer, 2008.
3 A. Bucciarelli, T. Ehrhard, and G. Manzonetto. Not enough points is enough. In *CSL 2007*, volume 4646 of *LNCS*. Springer Berlin, 2007.
4 V. Danos and T. Ehrhard. Probabilistic coherence spaces as a model of higher-order probabilistic computation. *Information and Computation*, 2011.
5 D. de Carvalho. Execution time of lambda-terms via denotational semantics and intersection types. *Math. Struct. in Comp. Sci.*, 2017.
6 U. de' Liguoro and A. Piperno. Nondeterministic extensions of untyped λ-calculus. *Information and Computation*, 1995.
7 A. Díaz-Caro. *Du typage vectoriel.* PhD thesis, Université de Grenoble, France, 2011.
8 T. Ehrhard. On Köthe sequence spaces and linear logic. *Math. Struct. in Comp. Sci.*, 2001.
9 T. Ehrhard. Finiteness spaces. *Math. Struct. in Comp. Sci.*, 15(4):615–646, 2005.
10 T. Ehrhard. A finiteness structure on resource terms. In *LICS 2010*. IEEE Computer Society, 2010.
11 T. Ehrhard and L. Regnier. The differential lambda-calculus. *Theor. Comp. Sci.*, 2003.
12 T. Ehrhard and L. Regnier. Böhm trees, Krivine's machine and the Taylor expansion of λ-terms. In *CiE 2006*, volume 3988 of *LNCS*. Springer, 2006.
13 T. Ehrhard and L. Regnier. Uniformity and the Taylor expansion of ordinary lambda-terms. *Theor. Comp. Sci.*, 2008.
14 J.-Y. Girard. The system F of variable types, fifteen years later. *Theor. Comp. Sci.*, 1986.
15 J.-Y. Girard. Normal functors, power series and lambda-calculus. *Annals of Pure and Applied Logic*, 1988.
16 R. Hasegawa. The generating functions of lambda terms. In *First Conference of the Centre for Discrete Mathematics and Theoretical Computer Science, Auckland, New Zealand, December, 9-13, 1996.* Springer-Verlag, Singapore, 1996.
17 A. Joyal. Foncteurs analytiques et espèces de structures. In *Proceedings of the "Colloque de combinatoire énumérative", Montréal, May 28 – June 1, 1985.* Springer, 1986.
18 J.R. Kennaway, J.W. Klop, M.R. Sleep, and F.J. de Vries. Infinitary lambda calculus. *Theor. Comp. Sci.*, 1997.
19 J. Laird. Fixed points in quantitative semantics. In *LICS 2016*. ACM, 2016.
20 J. Laird, G. Manzonetto, G. McCusker, and M. Pagani. Weighted relational models of typed lambda-calculi. In *LICS 2013*. IEEE Computer Society, 2013.
21 T. Leventis. *Probabilistic λ-theories.* PhD thesis, Aix-Marseille Université, 2016.
22 M. Pagani, C. Tasson, and L. Vaux. Strong normalizability as a finiteness structure via the taylor expansion of λ-terms. In *FOSSACS 2016*, volume 9634 of *LNCS*. Springer, 2016.
23 Takeshi Tsukada, Kazuyuki Asada, and Luke Ong. Generalised species of rigid resource terms. In Joel Ouaknine, editor, *Proceedings of the 32nd Annual IEEE Symposium on Logic in Computer Science, LICS 2017*, June 2017. To appear.
24 L. Vaux. The algebraic lambda calculus. *Math. Struct. in Comp. Sci.*, 2009.