Adaptive Notch Filters
for Prediction of Narrow Band Signals

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Abstract—This paper proposes efficient structures of Adaptive Notch Filters (ANF) implemented as constrained ARMA models. These ANF suit very well for fast frequencies tracking and evolutionary spectrum analysis for narrow band and sinusoidal signals in additive noise. Lastly, non-stationary ANF are considered and an estimation procedure implemented.

Index Terms—Adaptive Notch Filters, Narrow band Signals, Frequency Tracking, Spectrum Analysis, Non-stationary Signals, ANC.

I. INTRODUCTION

A. Objective

The main interest in this contribution, is the retrieval of the parameters of sinusoidal signal components (amplitudes, phases and frequencies), in noise. We assume (slowly) varying frequencies and varying amplitudes and phases. This is the case when the process is perturbed by the wind (cars, planes, flying robots, wind turbines ...) or the sea waves, swell (boats, submarine robots, wave turbines...). Other examples are the vibrations of processes in mechanics, hydraulics or pneumatics. It is often crucial to detect and predict the vibration signals in order to compensate or tackle their effects.

Recently introduced adaptive control techniques use compensation of the narrow band perturbation signals based on ANF [1], [2], [3], [4]. Autoregressive (AR) and ARMA (Autoregressive with Mean Average) modeling techniques have shown powerful potentialities. They are well suited for recursive signal spectral analysis, interesting for time varying signal spectrum [5], [6], [7].

However, when the noise level is too high and / or when the signal model has poles near or on the unit circle, serious (numerical or stability) problems arise [6].

B. Previous work

In the narrow-band signals case, successful applications have been realized with AR models [5], [8] but, they need high order autoregressive FIR filter. Very high order AR models has been shown to be asymptotically equivalent to an Adaptive Notch Filter (ANF). Additionally the noise effect is reduced by increasing the order of the AR filter. This adds the expense of computational cost [9]. This has been studied in extensive simulations [8], [10], [7].

Thus, the use of the a priori knowledge of a narrow-band feature of the signal, lead us to the design of Notch Filter Structure (NFS) with constrained poles and zeros. The Adaptive Notch Filters (ANF) have been proposed to provide the frequencies estimation and tracking [11] [9], [12]. These estimators can be used by implementation in a cascaded blocks of second order cells.

Some strategies has been proposed to track time varying parameters (like frequencies amplitudes and phases), which may be used in line [13]. ANF(s) are used to estimated the frequency components in a first step. Then, in a second step, after frequencies tracking the estimated frequencies are used to provide the estimation of amplitudes and phases. This can also be implemented in a recursive manner.

The proposed recursive algorithm consists in two steps. The first step involves a Recursive Maximum Likelihood algorithm to adapt the cascaded filters parameters [7], which will provide the frequencies estimates. The second step uses the last estimates, and then the estimations of amplitudes and phases are given by a Recursive Extended Least Squares (RELS) algorithm .

A theoretical analysis of these algorithms, have been presented in [6], [10] and the applicable estimation methods are reviewed in [10], [11] [14], [9], [15]. These last structures introduce in Parameter Adaptation Algorithms (PAA) some nonlinearity which may give some problems in the transient period of the estimation. The cascade implementation of second order sub models, have been claimed possible.

C. Contribution of this work

In this paper clarifies the rationale behind the Adaptive Notch Filters that we have proposed in the past and revisited in this paper. We give a complete theoretical analysis and prove the convergence of the two estimation algorithms:n one to estimate the frequency components by an ARML algorithm and the second to estimate the amplitudes and phases by an RLS algorithm with exponential weighting and a forgetting factor. These algorithms are applied for adaptive prediction of the swell perturbations on a boat.

In section II of this paper we present the Recursive estimation algorithms and the filter structure. We then develop an implementation of the cascaded ANF. The Prediction Error Method (PEM) is applied. This application leads to an Approximate Recursive Maximum Likelihood (ARML) algorithm for frequencies estimation. In a second step, using these estimates, a Recursive Least Squares (RLS) algorithm is employed for the estimation of amplitudes and phases.

The theoretical analysis of these algorithms and their performance evaluation is presented in section III.

Section for will be dedicated to effective Prediction of narrow band signals and parameters estimation. The narrow

band signals, considered are the sea waves. After the ANF for the signal Prediction, we estimate the frequencies and the deduce the estimations of the amplitudes and phase. The algorithms Validation is done by using the swell signals perturbing a boat motion.

II. RECURSIVE ESTIMATION ALGORITHMS AND FILTER STRUCTURE

The proposed recursive estimation procedure consists of two stages, as the parameters to estimated are in non linear combination. The first step processed involves an ANF structure implemented in cascaded form for the estimation of the signal frequencies. An Approximate Recursive Maximum Likelihood (ARML) method is developed and used. The second step uses Recursive Instrumental Variables (RIV) to avoid non linearities and get an appropriate model for linear prediction techniques. This leads us a model linear in its parameters (to be estimated). An adaptive estimation algorithm, based on Weighted Recursive Least Squares (WRLS), is used to provide the amplitudes and phases estimations.

The involved signals may be modeled as follows where v_k is the noise disturbance, $C_i(k)$ and $\beta_i(k)$ are, respectively, the time varying amplitude and phase of the frequency component $f_i(k)$. All the parameters are susceptible to vary in time. T_e is the sampling period.

$$y_k = \sum_{i=1}^{p} C_i(k).Sin(2.\pi f_i.T_e.k + \beta_i(k)) + v_k \quad (1)$$

A. Amplitudes and Phases Estimation with RIV-WRLS

In this step, we assume that the frequencies of the signal are known (well estimated by the proposed ARLM algorithm) and we develop an estimator for the amplitudes and phases of each frequency component. We use a Weighted Recursive Least Squares algorithm, as in [11], to estimate the amplitude and phase of each sinusoid. The frequencies estimations are provided by the ANF, as will be developed in the next subsection in a previous step.

The signal Model of equation (1), can be decomposed in Fourier Series. It may be written as linear combination of the Fourier basis elements (which are the Instrumental Variables (IV) to be used in estimation):

$$y_k = \sum_{l=1}^{p} (g_l Cos(2\pi f_l T_e.k + h_l Sin(2\pi f_l T_e.k)) + v_k \quad (2)$$

where the amplitudes C_l and phases β_l for $(l = 1...p)$ are given from the Fourier series parameters by

$$C_l = \sqrt{g_l^2 + h_l^2} \quad and \quad tan(\beta_l) = g_l/h_l \quad (3)$$

It is worthwhile to note that the Fourier series parameters are identifiable. We can then define the parameter $\hat{\theta}(k)$ and the regression (by Recursive Instrumental Variable RIV) $\hat{\psi}(k)$

vectors as follows

$$\begin{cases} \hat{\theta}(k) = [g_1, g_2, ..., g_p, h_1, h_2, ..., h_p]^T \\ \Phi_k = [C, S]^T \ with \\ C = [Cos(2\pi f_l.T_e.k) \ .., \ Cos(2\pi f_p.T_e.k)] \\ S = [Sin(2\pi f_l.T_e.k) \ ,..., \ Sin(2\pi f_p.T_e.k)] \end{cases} \quad (4)$$

The parameters are then estimated using an WRLS algorithm as follows:

$$\begin{cases} y_k^0 = \hat{\theta}(k-1)^T.\Phi_k \\ \varepsilon^0(k) = y_k - \hat{\theta}(k-1)^T.\Phi_k \\ G_k = \frac{1}{\lambda_0} \left(G_{k-1} - \frac{G_{k-1}\Phi_k\Phi_k^T G_{k-1}}{\lambda_0 + \Phi_k^T G_{k-1}\Phi_k} \right) \\ \hat{\theta}(k) = \hat{\theta}(k-1) + G_k.\Phi_k.\varepsilon^0(k) \\ C_l = \sqrt{g_l^2 + h_l^2} \quad tan(\beta_l) = g_l/h_l \end{cases} \quad (5)$$

where y_k^0 is the a priori prediction and $\epsilon^0(k)$ is the a priori prediction error. In equation (5) filtered versions of the signal y_k can be used; For example, we can use the prediction (of the ANF) produced by the frequency estimation stage. Doing this we will get the RIV coupled with the signal Output Prediction Error Method (RIV-OPEM).

We can note, also, that the RIV regression vector will verify the Persistent Excitation Condition (PEC). The use of a constant λ_0 as an exponential forgetting factor will help for the alertness of the amplitudes and phases estimation by WRLS. The adaptation gain G_k will not decrease to zero and remain positive. Then estimation will remain correct in case of slowly time varying parameters.

B. Frequency estimation whit cascaded ANF

Adaptive Notch Filters are very well suited for estimation of the sinusoidal component frequencies. The ith sinusoidal component will be filtered by the ith second order ANF:

$$H_i(z^{-1}) = \frac{1 + a_i z^{-1} + z^{-2}}{1 + r.a_i z^{-1} + r^2 z^{-2}} \quad (6)$$

The last equation presents a second order notch filter. Let us note that the transfer function does not depend on amplitudes and phases. Those informations are in the initial conditions. The notch bandwidth is defined by the parameter 0<r<1. The parameter a_i depends on the notch frequency

$$a_i = -2cos(2\pi f_i T_e) \quad (7)$$

(for $i = 1...p$). As they are p sinusoidal components in equation (1), let us consider the notch filter transfer function as p cascaded second order Notch cells:

$$H(z^{-1}) = \prod_{i=1}^{p} H_i(z^{-1}) \quad (8)$$

With no loss of generality we assume identical bandwidth parameter r for the notches. Thus independence between frequencies yields independence between the parameters of each second order cell. Then each cell can be adapted independently of the others after pre-filtering the signal by the others. For comb filter, relations between the frequencies can be used

to reduce the computational burden. If all the frequency components are independent we can write

$$\widetilde{y}_k^j = \prod_{\substack{i=1 \\ i \neq j}}^{p} H_i(q^{-1}) y_k \qquad (9)$$

If we suppose that the cascaded filters $H_i(z^{-1})$ with central frequency f_i (for $i = l...p$ and $i \neq j$) have converged then the filter $H_j(z^{-1})$ will remove one of the jth sinusoidal components, remaining after the filters $H_i(z^{-1})$, for $i = 1...p$ and $i \neq j$.

$$\widetilde{y}_k^j = \prod_{\substack{i=1 \\ i \neq j}}^{p} \frac{1 + a_i q^{-1} + q^{-2}}{1 + r.a_i q^{-1} + r^2 q^{-2}} . y_k \qquad (10)$$

This means that in the signal $\widetilde{y}_k^j$ the p-1 sinusoidal components have been removed by the corresponding notches and then the prediction error ε_k is obtained by removing the jth one:

$$\varepsilon_k = H_j(q^{-1}) \widetilde{y}_k^j = \frac{1 + a_j q^{-1} + q^{-2}}{1 + r.a_j q^{-1} + r^2 q^{-2}} \widetilde{y}_k^j \qquad (11)$$

Thus, there will exit a unique global minimum for the Mean Squared Error (MSE) Criterion, as each notch remove one frequency component. Following the approach of the Output Prediction Error Method (OPEM), for the estimation algorithm, we obtain as the estimation gradient:

$$\psi_{k-1}^j = -\frac{d\varepsilon_k}{da_j} = \frac{(1-r)(1-rq^{-2})}{1 + r.a_j q^{-1} + r^2 q^{-2}} . \widetilde{y}_{k-1}^j \qquad (12)$$

The RML algorithm may be summarized by the following equations, where F_k^j is the adaptation gain and $0 < \lambda < 1$ the forgetting factor:

$$\begin{cases} for \ \ j = 1, ..., p, \ \ do \\ \hat{a}_k^j = \hat{a}_{k-1}^j + F_{k-1}^j . \psi_{k-1}^j . \varepsilon_k \\ F_k^j = F_{k-1}^j / (\lambda + \psi_{k-1}^j . F_{k-1}^j . \psi_{k-1}^j) \end{cases} \qquad (13)$$

The computational complexity of the RML algorithm comes essentially from the computation of the gradient and the successive filtering (equations 10,11,12). A first approximation can be done to reduce computational complexity. It yields an Approximate RML (ARML) algorithm. We can then use the formula (using equation II-B instead of 12):

$$\psi_{k-1}^j \simeq \frac{s_{k-1}^j - r\varepsilon_{k-1}}{1 + r.a_j q^{-1} + r^2 q^{-2}} . \widetilde{y}_k^j \qquad (14)$$

with the signals s_k^j computed as follows

$$\begin{cases} for j = (p-1), ..., 1 do \\ y_k^j = \prod_{l=j+1}^{p} H_l(q^{-1}) . y_k \\ s_k^j = y_k^j - y_k^{j-1} \end{cases} \qquad (15)$$

Another suggestion is to adapt the Notch filter second order cells successively for $i = 1, .., p$ in a cascaded way or recur-

sively. Each cell get as input the output error computed from the previous cells only, in a recursive manner. The bandwidth or what we call the de-biasing parameter r_k must be a positive. In our experience we choose this parameter exponentially time varying, from 0 or r_0 to one (or to r_f), according to the following expression: $r_k = r_d.r_{k-1} + (1 - r_d).r_f$.

III. ANALYSIS AND PERFORMANCE EVALUATION

A. Convergence of the ANF (OPEM RML)

1) Second order ANF with an RML algorithm: Let us first consider the case of a single cell $H_j(z^{-1})$ (see (6)) driven by the signal defined in (1) with $p = 1$.

Assume, either if $p > 1$, that the preceding filters $H_i(z^{-1})$ (for $i = 1.. \ j-1, j+1, .. \ n$ and $i \neq j$) have, all of them, converged to their optimal value. The bandwidth, of the filter $H_i(z^{-1})$ are assumed large enough to remove (or attenuate) the other frequencies (f_i respectively $i \neq j$).

Thus $\tilde{y}_k^j$ is composed by a single frequency f_j plus an additive independent noise η_k.

$$\tilde{y}_k^j = C_j(k).Sin(2.\pi f_j.T_e.k + \beta_j(k)) + \eta_k \qquad (16)$$

Tacking for the signal an AR model with two complex conjugate poles on the unit circle and applying the RML Parameter Estimation algorithm, the a posteriori error can be written:

$$\varepsilon_k = \frac{\hat{A}_j(rq^{-1})}{A_j(rq^{-1})} . (\hat{a}_j - a_j) . \psi_{k-1}^j + \frac{\eta_k}{A_j(rq^{-1})} \qquad (17)$$

with

$$\begin{aligned} \tilde{y}_k^j &= -a_j \tilde{y}_{k-1}^j - \tilde{y}_{k-2}^j + \eta_k \\ A_j(rq^{-1}) &= 1 + r.a_j q^{-1} + r^2 q^{-2} \\ \hat{A}_j(rq^{-1}) &= 1 + r.\hat{a}_j q^{-1} + r^2 q^{-2} \end{aligned} \qquad (18)$$

It is well known, in the general case, that the RML algorithm needs a stability monitoring. For the ANF this procedure can be removed in virtue of the following Lemma. We use a time varying de-biasing parameter.

Lemma 1. *Let r_k be a positive (bandwidth) parameter exponentially time varying, from 0 to one, according to the following expression:*

$$r_k = r_d.r_{k-1} + (1 - r_d).r_f \qquad (19)$$

.

Then there exit positive constants, less than 1, (r_0, r_d, r_f) such that $\hat{A}(r_k, q^{-1})$ is infinitely often stable.

Proof: The bandwidth (or de-biasing) parameter r_k acts like a contraction of the poles, attracting them toward the center of the unit circle. In addition r_k starts at r_0, for k=0, and goes to r_f, when k goes to infinity. We can then take r_f less than the inverse of the maximum modulus of the unstable poles of $\hat{A}(r_k, q^{-1})$. ∎

Now applying the theoretical background, on convergence of adaptation algorithms, presented in [16] and particularly the Theorem 4 of [16] leads to the following Theorem to establish the convergence of the ANF OPRM RML algorithm.

Theorem 2. *: Under the following assumptions:*

- *$\hat{A}(r_k, q^{-1})$ is infinitely often stable ($r_k.\hat{a}_j(k) < 2$)*
- *for some $\hat{a}_j$ fixed ψ_k^j , ε_k and $\frac{\eta_k}{A_j(rq^{-1})}$ are quasi stationary signal.*
- *η_k is independent of ψ_{k-1}^j and*
- *the transfer function $\frac{\hat{A}(rz^{-1})}{A(rz^{-1})} - \frac{1}{2}$ is SPR (Strictly Positive Real).*

Then we get, for the estimation algorithm, the following:

- *$D_c = \{a$ such as $E\{(\hat{a}_j - a_j).\psi_{k-1}^j\} = 0\}$ is the convergence domain*
- *$\lim(\epsilon_k - \eta_k) = 0$ WP1 (With probability one)*
- *$\lim(\hat{a}_j(k))\epsilon D_c$ WP1 (With probability one)*

To ensure the SPR condition we consider the following lemma.

Lemma 3. *: There always exist an r such that $\frac{A(r.\hat{q}^{-1})}{A(r.q^{-1})}$ is a Strictly Positive Real transfer function (SPR).*

The last condition on r is less restrictive than the one for the stability. We can then conclude that the Lemmas 1 and 2, with the last theorem, allow us to assert the global stability of the ANF if r is appropriately chosen such that $\hat{A}(r.q^{-l})$ is always stable during estimation progress.

2) Second order ANF with an ELS algorithm: If the ELS estimation method is applied, the estimation error becomes [13]:

$$\varepsilon_k = \frac{1}{A_j(rq^{-1})}.(\hat{a}_j - a_j).\psi_{k-1}^j + \eta_k \qquad (20)$$

There exits an upper limit l for the de-biasing parameter in order to get the SPR condition. This limit l depends on the signal frequency and limits the accuracy of the estimation. The SPR condition on r is more restrictive than the stability one. Then Theorem 2 cannot be applied in all cases.

Thus the local stability cannot be established although the global stability can be ensured.

3) Uniqueness of the RML estimates: In theorem 2 we have got asymptotically WPl: $\varepsilon_k = \frac{\eta_k}{A_j(rq^{-1})}$

Equations 4 and 14 yield

$$\varepsilon_k = \frac{\hat{A}_j(q^{-1})}{\hat{A}_j(rq^{-1})}.\tilde{y}_k^j = \frac{\hat{A}_j(q^{-1})}{\hat{A}_j(rq^{-1})}.\frac{\eta_k}{A_j(q^{-1})} \qquad (21)$$

It then follows from 17 and 18 that:

$$\frac{\hat{A}_j(q^{-1})}{\hat{A}_j(rq^{-1})} = \frac{A_j(q^{-1})}{A_j(rq^{-1})} \qquad (22)$$

and then, $\hat{a}_1 = a_1$ with probability one (WP1) if $r \neq 1$.

Theorem 4. *: : If the order is correctly chosen (the number of the frequency components is matched), then the RML estimates for ANF are unique*

The main results concerning ANF are then:

1) The global convergence can be ensured with an appropriate choice of r the de-biasing parameter without any stability monitoring.
2) The RML estimates are unique.

For p cascaded cells adapted along the same procedure, each cell will converge near to a local minimum. The filtering by ANFs will make these minima distinct, removing, for each cell, one of the other frequencies. This will not be the case if the implementation of the Notch cells is made in parallel instead of in cascaded form.

B. Convergence of Amplitudes and phases estimation, RIV-WRLS

The Exponential Convergence of the estimation of the amplitudes and the phases is guaranteed by the WRLS algorithm properties, as shown in the following theorem which uses the concept of persistent excitation see eg [5], [7].

Theorem 5. *Convergence of the WRLS algorithm*

The WRLS estimation scheme of equations (II-A) with an exponential forgetting factor λ is exponentially stable, under persistent excitation condition.

Proof. The proof of this theorem can be conducted in the same lines as in [17].

The regression vectors (RIV) are composed by p sinusoids having different frequencies and phases. Then the Persistent Excitation Condition (PEC) is satisfied.

Equation (23) yields by lemma 1 of [17]. The regressors are composed by p sinusoids having different frequencies. Then the persistent excitation condition is satisfied:

$$0 < \gamma_1.I < \sum_{k=j}^{j+s} \Phi_k \Phi_k^T < \gamma_2.I < \infty \qquad (23)$$

Equation (23) yields by lemma 1 of [17]:

$$0 < \frac{\gamma_1.(\lambda^{-1} - 1)}{(\lambda^{-(s+1)} - 1)}.I < F_{k-1}^{-1} < \frac{\gamma_2}{1 - \lambda^{s+1}}.I + o(\lambda^k) \quad (24)$$

The adaptation gain is bounded and will never be zero (the alertness property of WRLS and its tracking capabilities). Finally reasoning as in [17] we obtain:

$$\left\|\tilde{\theta}_k\right\|^2 < \frac{\lambda^{s+1} - 1}{\gamma_1(\lambda^{-1} - 1)}.\lambda^k.\lambda_{max}(F_0^{-1}).\left\|\tilde{\theta}_0\right\|^2 \qquad (25)$$

$$\square$$

This means that the parameter estimation error will be always strictly decreasing. Then the convergence of the estimations.

Comparative Simulation results will be presented in the final version of this paper.

IV. SIMULATIONS AND EXPERIMENTAL VALIDATION

In order to test the proposed ARML and RLS for predicting narrow band signals, we consider simulation of non-stationary signal with 3 frequency components and an additive centered and normal white noise v_k (with variance $\sigma_v^2 = 1$) and signal to noise ratio SNR = 25dB.

$$y_k = \sum_{i=1}^{3} C_i(k).Sin(2.\pi f_i.T_e.k + \beta_i(k)) + v_k \qquad (26)$$

Frequencies, amplitudes and phases of the three components are slowly time varying to prove the tracking capability of the estimation. The sampling rate is $F_e = 2Hz$ and the 3 frequencies are set initially to $f_1 = 0.2Hz$, $f_2 = 0.3Hz$, $f_3 = 0.4Hz$ and vary as in Figure (13).
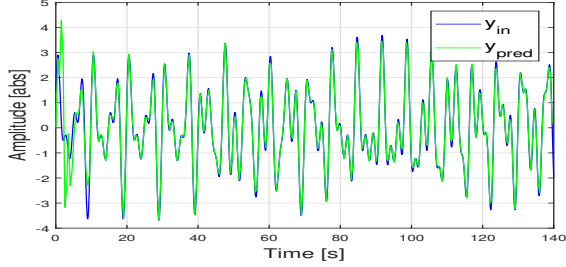

Fig. 1: Signal (blue) and its ANF prediction (green)

A. Frequencies estimation by cascaded ANF and ARML

The first stage estimates the three cascaded Notch Filter parameters. The parameters estimates $\hat{a}_i(0)$ are set to zero at the beginning. All ANF bandwidths are initially set at $r_0 = 0.1$ and exponentially increases to $r_f = 0.8$ with a factor of $r_d = 0.99$ (see equation 19). The figure 2 depicts the evolution of the ANF parameters estimation $\hat{a}_i(k)$ versus time ($t = kT_e$). The dotted lines represent the true values of these parameters. The convergence is fast and estimations are precise and accurate. This results in a good (one step ahead) prediction of the signal as shown as in Figure (1).
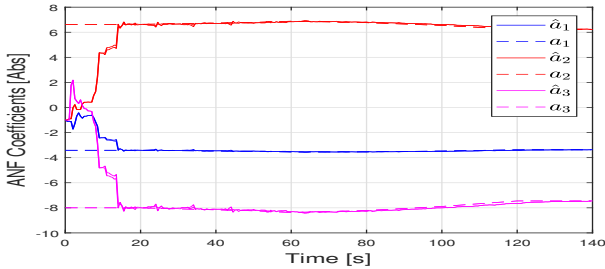

Fig. 2: Notch Filters parameters estimates versus time

The frequencies deduced, on line, from the estimated parameters using equations 7. The following figure (3) illustrates the good convergence of the on line frequency estimations, after a short transient period (less than 20s). We observe that the frequency estimation converges after 36 sampling points (18s) and tracks simultaneous linear variations of the instantaneous frequencies.

B. Amplitude and phase estimation by WRLS

Using previous estimated frequencies, this second stage estimates amplitudes C_i and phases β_i of each one of the three components, using the proposed Weighted Recursive Least Squares algorithm. The regression vector and the parameters vector to be estimated are defined in the equation (4) and the setimations are given by equations (5 and II-A).
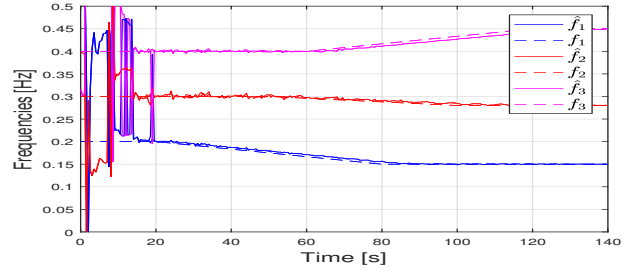

Fig. 3: Frequencies estimates versus time

The initial value of the adaptation gain is set to $G_0 = 100$ and the forgetting factor lambda λ_0 is set to 0.99. The parameters vector $\hat{\theta}$ is initially set to 0. Components of $\hat{\theta}$: g_l and h_l are used to compute C_i and β_i using the last equation of 5.

The following figures illustrate the convergence of WRLS for amplitude/phase estimation. The figure 4 depicts the evolution of components amplitudes estimates versus time.
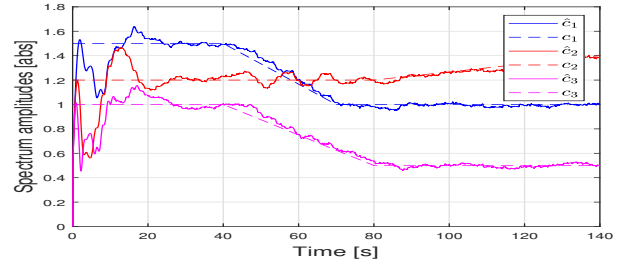

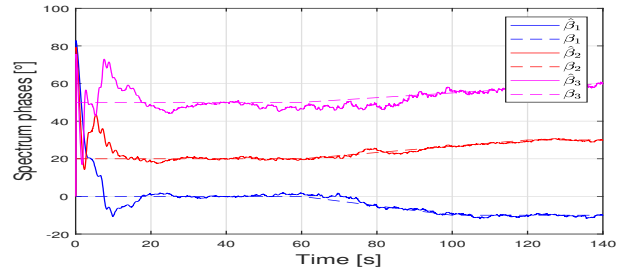Fig. 4: Amplitudes estimates versus time

Fig. 5: Phases estimates versus time

The figure 4 shows the evolution of components phases estimates versus time. We observe that the amplitude/phase estimation converges after 50 points (25s) and tracks simultaneous linear variations of amplitudes and phases. The last figure shows that the prediction is quite good for a signal with 6 frequency components.

V. CONCLUSION

The main interest of the ANF in Cascade form is to simplify computation of the estimated frequencies and in case of independently time varying frequencies. A second order filter model appears to be faster to track variation than a higher order one.

The main results in this paper demonstrate the good performances observed in simulation for the ANF. This confirm the theoretical analysis and convergence proves (for RML and ELS PAAs). The exponential convergence of the estimation of the amplitudes and phases has been also proved.
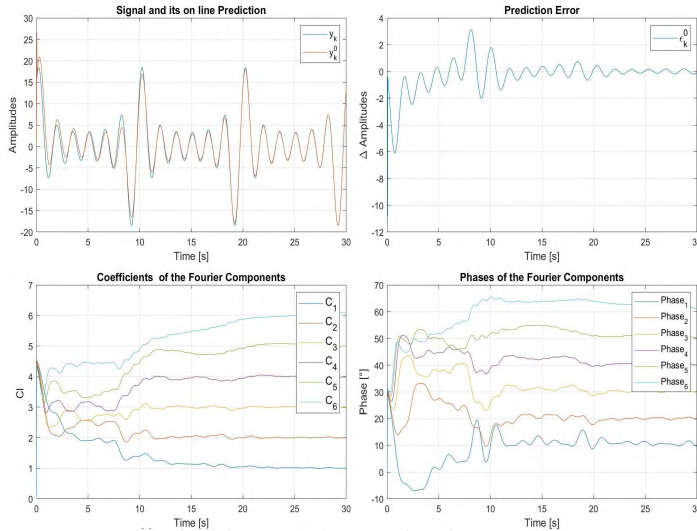
Fig. 6: Phases estimates for 6 components

The proposed (OPEM-RML) algorithm is asymptotically consistent and robust faced with the neglected dynamics and noise. In addition, for case of time varying signals, its tracking capabilities insure very good estimations and tracking of the parameters time variation.

Several estimation algorithms may be applied with these structures, like RELS, ARML, RIV (Recursive Extended Least Squares, Approximate Recursive Maximum Likelihood, Recursive Instrumental Variables with WRLS). Some special procedures may be employed to increase the convergence rate (by adjusting the bandwidth parameter r) and to enhance the estimation consistency and reduce the algorithm complexity.

The obtained estimation accuracy is better with the RML Adaptation Algorithm than with the Recursive Extended Least Squares method for which an upper limit of the de-biasing parameter is crucial in order to have a convergence without local instabilities.

The exponential convergence of the RIV-WRLS estimates of amplitudes and phases is proved by use of the theoretical background on persistent excitation. These results are very important for the time varying systems or in case of neglected dynamics due to the resulting robustness of the proposed algorithms. The implementation form studied here is computationally attractive and robust also when frequency, amplitude or phase are time varying.

Acknowledgment

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