function "$W(s)\Delta$" where Δ is an unknown constant and $|\Delta| < 1$ represents for the un-modeled dynamics.

The paper investigates the position control of electrical motor drives that can be configured as structure of Fig. 1. This problem is formulated as follows. Consider the basic structure of a motor drive system and assume that

1. The frequency response from voltage *u* to rotor position *θ* is available.
2. Δ is an unknown constant and $|\Delta| < 1$.

Under these assumptions the control objective is to calculate the family of PID controllers that stabilize the uncertain plants $P(s)(1 + W(s)\Delta)$ where $|\Delta| < 1$ and satisfies H_∞-norm consideration on complementary sensitivity function in the presence of un-modeled dynamics.

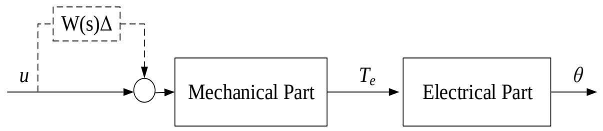

Fig. 1. Motor drive **system with un-modeled dynamics**

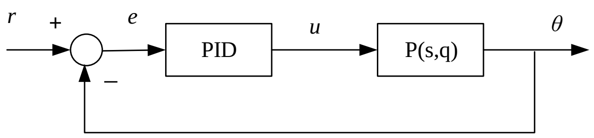

Fig. 2. Feedback structure with uncertain plant and PID

Fig. 2 shows the feedback system with PID controller in which $P(s,q)$ is an uncertain plant and q can be substituted by Δ. In the next section, a survey on the algorithm proposed for nominal stability in [11] and its generalization to uncertain plants is presented and it is illustrated that the control objectives of this paper on motor drive system can be satisfied in a same way.

III. PID CONTROLLER DESIGN: MODEL FREE APPROCH

In this section, the problem of achieving the family of stabilizing PID controllers for nominal stable plant $P(s)$ proposed in [11] is reviewed. Then by verifying a theorem, this approach is generalized to plants with an uncertain parameter. Similar approach is proposed for unstable plants which are omitted here because of room limitation.

The feedback system with PID controller is shown in Fig. 2. First, some mathematical preliminaries are introduced. Consider a real rational function

$$P(s) = \frac{A(s)}{B(s)}$$

where $A(s)$and $B(s)$ are polynomials with real coefficients and of degrees *m* and *n*, respectively. Assume that $A(s)$ and $B(s)$ have no zero on $j\omega$ axis. Let $z^+, p^+ (z^-, p^-)$ determine the number of open right halp plane (RHP) (open left half plane (LHP)) zeros and poles of$P(s)$. Also let $\Delta|_0^\infty(P(j\omega))$ denotes the net change in phase of $P(s)$ as ω runs from *0* to $+\infty$. Then

$$\Delta|_0^\infty(P) = \frac{\pi}{2}[z^- - z^+ - (p^- - p^+)] \quad (1)$$

Define the (Hurwitz) signature of $P(s)$ as

$$\sigma(P) = \frac{2}{\pi}\Delta|_0^\infty(P(j\omega)) \quad (2)$$

and since $P(s)$ has no pole and zero on $j\omega$ axis, it can be written

$$\sigma(P) = -(n - m) - 2(z^+ - p^+) \quad (3)$$

The value of $n - m$ can be calculated from the frequency data of $P(s)$ and z^+ can be calculated from (3). Let

$$P(j\omega) = P_r(j\omega) + jP_i(j\omega)$$

where $P_r(j\omega)$ and $P_i(j\omega)$ denote the real and imaginary parts of $P(j\omega)$, respectively. Assume that the real, distinct, finite zeros of $P_i(j\omega) = 0$ denote as $\omega_0, \omega_1, \dots, \omega_{l-1}$ such that

$$0 = \omega_0 < \omega_1 < \cdots < \omega_{l-1}$$

and consider the modified PID controller as

$$C(s) = \frac{k_i + k_p s + k_d s^2}{s(1 + sT)}$$

where k_p, k_i and$k_d$are the proportional, integral and derivative coefficients of PID controller and *T* is a positive constant.

Lemma 1: *Let*

$$F(s) := s(1 + sT) + (k_i + k_p s + k_d s^2)P(s)$$

and

$\overline{F}(s) = F(s)P(-s).$

Then the closed loop stability is equal to

$$\sigma\left(\overline{F}(s)\right) = n - m + 2z^+ + 2.$$

Let

$$\overline{F}(j\omega) = \overline{F}_r(\omega, k_i, k_d) + j\omega\overline{F}_i(\omega, k_p)$$

where

$$\overline{F}_r(\omega, k_i, k_d) = (k_i - k_d\omega^2)|P(j\omega)|^2 - \omega^2 T P_r(\omega) + \omega P_i(\omega)$$

and

$$\overline{F}_i(\omega, k_p) = k_p|P(j\omega)|^2 + P_r(\omega) + \omega T P_i(\omega)$$

Consider $\overline{F}_i(\omega, k_p) = 0$ and define

$$k_p = -\frac{\cos\phi(\omega) + \omega T \sin\phi(\omega)}{|P(j\omega)|} := g(\omega) \quad (4)$$

and $J = sgn[\overline{F}_i(\infty^-, k_p)]$ where $k_p^{min} < k_p^* < k_p^{max}$.

Theorem 1: *Let* $\omega_1 < \omega_2 < \cdots < \omega_{l-1}$*denote the distinct frequencies of odd multiplicities which are solutions of*

$$\overline{F}_i(\omega, k_p^*) = 0.$$

Determine strings of integers $I = [i_0, i_1, i_2, \dots, i_l]$ *where* $i_t \epsilon \{-1,1\}$ *such that:*

$$\frac{dg}{dk} = -\frac{(\frac{dP_r}{dk} + \omega\frac{dP_i}{dk})(p_i^2 + P_r^2) - 2\left(P_r\frac{dP_r}{dk} + P_i\frac{dP_i}{dk}\right)(P_r + P_i\omega)}{p_i^2 + P_r^2}$$

Without loss of generality, consider the case of monotonic increasing in $g(\omega)$. Increasing the uncertain parameter leads to increasing in $g(\omega)$ if the following inequality holds:

$$(P_r^2 - P_i^2 + 2\omega P_r P_i)dP_r + (P_i^2\omega - P_r^2\omega + 2P_rP_i)dP_i > 0 \quad (11)$$

Assumption 1: Let

$$P_i^2\omega - P_r^2\omega + 2P_rP_i > 0. \quad (12)$$

and note that the symbol '>' could be inversed; but for simplicity it is assumed as '>'. Finally, consider the real rational symbol of (12). So (11) can be transformed to

$$\frac{dP_i}{dP_r} > \frac{(P_i^2 - P_r^2 - 2\omega P_r P_i)}{(P_i^2\omega - P_r^2\omega + 2P_rP_i)} \quad (13)$$

and by change of variable as:

$$z = \frac{P_i}{P_r}$$

the inequality (13) can be changed to:

$$P_r dz > \frac{-z^3\omega - z^2 - z\omega - 1}{z^2\omega - \omega + 2z} dP_r. \quad (14)$$

Assumption 2: Let

$$P_r > 0 \quad (15)$$

and

$$\frac{-z^3\omega - z^2 - z\omega - 1}{z^2\omega - \omega + 2z} > 0 \quad (16)$$

similar to Assumption 1, the real rational symbol of (15) and (16) will be considered finally. From (15) and (16), the inequality (14) leads to:

$$\ln\left(\frac{-z\omega - 1}{(z^2+1)P_r}\right) > 0 \quad \Rightarrow \quad \frac{-z\omega - 1}{(z^2+1)P_r} > 1$$

and since $z = \frac{P_i}{P_r}$, then

$$p_i^2 + P_r^2 + P_i\omega + P_r < 0$$

that is one of the necessary conditions imply that increasing the uncertain parameter q leads to increasing in $g(\omega)$. Assumptions 1 and 2 imply another necessary condition as

$$-(P_r + P_i\omega) > 0.$$

Remark 2: Bode diagrams corresponding to $P(j\omega, q_{min})$ and $P(j\omega, q_{max})$ could be recognized from the frequency responses set of uncertain plant $P(j\omega, q)$. In fact if (9) or (10) holds, then from monotonic variation of the function $g(\omega)$, it could be deduced that the uppermost and lowermost plots of $g(\omega)$ are corresponding to $P(j\omega, q_{min})$ and $P(j\omega, q_{max})$, not necessarily respectively.

The following Corollary shows that the problem can be solved easier when the uncertain parameter is in the feedforward path of control loop.

Corollary 1. *Let $P(s,q)$ be a real rational function for an uncertain LTI plant and $q \in \{q_{\min}, q_{\max}\}$ is an uncertain parameter that appears in the feedforward path. Then the family of stabilizing PID controllers for uncertain plant is a subset of common set between two set of stabilizing PID parameters for $P(s, q_{\min})$ and $P(s, q_{\max})$ if one of the below sets satisfy*

$$\begin{cases} (-P_r + \omega P)_i > 0 & : \ if \quad g(\omega) \quad is \quad ascendant \\ (-P_r + \omega P)_i < 0 & : \ if \quad g(\omega) \quad is \quad decendant \end{cases}$$

(17)

for $\omega \geq 0$ and for only one value of uncertain parameter q.

proof: It can be written

$$P(j\omega, q) = qP_r(j\omega) + qP_i(j\omega)$$

and

$$g(\omega) = -\frac{qP_r(j\omega) + \omega qP_i(j\omega)}{q^2(P_r(j\omega) + P_i(j\omega))}.$$

Then

$$\frac{dg}{dq} = \frac{1}{q^2} \cdot \frac{P_r + \omega P_i}{P_r^2 + P_i^2}$$

So $g(\omega)$ is monotonic if (17) satisfies.

The procedure for calculating the family of stabilizing PID controllers is summarized in the below algorithm.

Algorithm 2.

1. Determine $P(j\omega, q_{min})$ and $P(j\omega, q_{max})$ for $\omega \geq 0$ from $g(\omega)$ plots using Remark 2.
2. Using Algorithm 1, calculate the family of stabilizing PID controllers for two plants $P(s, q_{min})$ and $P(s, q_{max})$. Determine the common space of PID parameters between two calculated family. The family of stabilizing PID controllers for uncertain plant $P(s,q)$ is a subspace of this common space.
3. Calculate the exact family of stabilizing PID controllers using the approach proposed in Remark 1.

Now it can be shown that the problem of satisfying some performance specifications for uncertain plant $P(s,q)$ could be transformed to problem of robust stabilizing the uncertain plant $P(s,q)$ with additional virtual uncertain parameter.

Many performance attainment problems for uncertain plant $P(s,q)$ can be cast as the simultaneously stabilization of the uncertain plant and the family of real and complex plants [11]. For example an H_∞-norm achievment on the complementary sensitivity function, that is $\|W(s)T(s)\| < \gamma$, is equivalent to simultaneously stabilizing the uncertain plant $P^C(s,q,\theta)$ as

$$P^C(s,q,\theta) = \{\left(1+\frac{1}{\gamma}e^{j\theta}W(s)\right)P(s,q) : \theta\epsilon[\theta_{min},\theta_{max}]\} \quad (18)$$

where θ is a virtual uncertain parameter and $W(s)$ is a weight selected by designer.

Let the family of stabilizing PID parameters for uncertain plant $P(s,q)$ can be calculated from Algorithm 1 and Let $P^C(s,q,\theta)$in (18) be a real rational function. If $g(\omega)$ corresponding to (18), varies monotonically by monotonic varying the virtual uncertain parameter θ, for any q and any θ, then the family of stabilizing PID controllers for uncertain plant $P(s,q)$ that satisfy the H_∞-norm specification on the complementary sensitivity function, is a subspace of PID controllers obtained from simultaneously stabilization of two plants

$$P^C(s,q_{min}) = \left[1+\frac{1}{\gamma}W(s)\right]P \quad (19)$$

$$P^C(s,q_{max}) = \left[1+\frac{1}{\gamma}W(s)\right]P(s,q_{max}) \quad (20)$$

and the exact family can be obtained using the approach proposed in Remark 1.

IV. A REVIEW ON THE APPLICATIONS OF THE PROPOSED APPROACH

In this section, some applications of the proposed method are presented. In fact, it is illustrated that some control objectives can be cast as robust stabilizing of a plant with an uncertain parameter. For example:

Example 1: Performance achievement

Many performance achievement problems for uncertain plant $P(s,q)$ can be cast as the simultaneously stabilization of the uncertain plant and the family of real and complex plants. Some of these performance achievement problems for nominal plant are listed in [11]; for example, the problem of H_∞-norm achievement on the complementary sensitivity function is equivalent to simultaneously stabilizing the plant P(s) and the family of real plants

$$P^C(s,q,\theta) : q\in\{q_{\min},q_{\max}\}, \theta\in\{0,2\pi\}$$

and the exact family can be obtained using some test points. The other specifications such as H_∞-norm achievement on the sensitivity function and phase margin can be satisfied by the same approach.

Example 2: Robustness against loss of effectiveness in actuator

Fig. 3 Shows the feedback structure of a plant with PID faced to loss of effectiveness in actuator where L is the parameter corresponding to loss of effectiveness and belongs to $(0,1]$. Obviously this structure can be cast as a plant with an uncertain parameter, i.e., $P(j\omega,L)$ with PID where $L=q$ is the uncertain parameter. There is similar case when loss of effectiveness happens in sensors. Thus the family of robust PID controllers for plants faced to loss of effectiveness can be calculated by the approach proposed in the previous section. For this special case, the problem of controller synthesis could be handled easily using Corollary 1.

Example 3: Robustness in plants with a dominant uncertain parameter

Some plants that are faced to parameter variations can be approximated by plants with a dominant uncertain parameter. Obviously the proposed approach can be applied in this case to calculate the family of robust stabilizing PID controller.

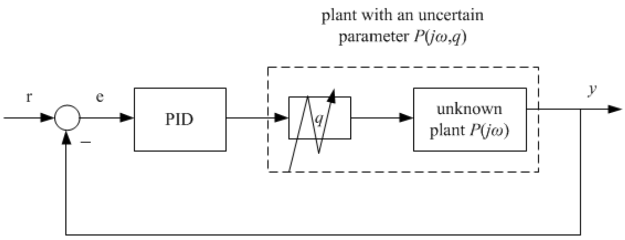

Fig. 3. The feedback structure of plant faced to loss of actuator effectiveness with PID

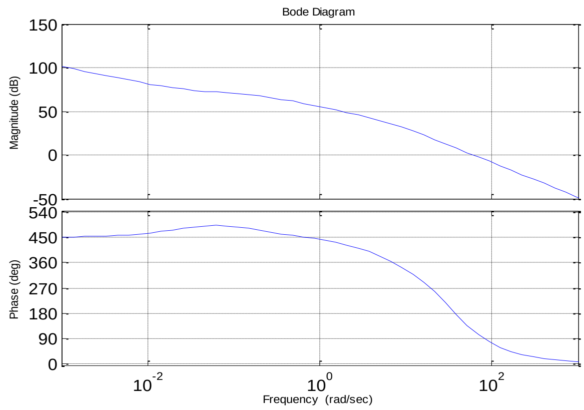

Fig. 4. The frequency response of motor drive system

V. SIMULATION ON INDUCTION MOTOR

The model studied in this section is an induction motor drive system introduced in [9, 12] that has the similar structure to Fig. 1 in [14] and its nominal frequency response is shown in Fig. 4. The frequency response of such systems can be obtained by virtual sine sweeping [13]. The corresponding plot of $g(\omega)$ is shown in Fig. 5. Since $R\geq 3$ (from (8)), the admissible range of k_p is $[-25,395]$. Calculating the(k_i,k_d)values for $k_p=50$ and *T=1* is resulted to the following inequalities

$$\begin{cases} k_i > 0 \\ k_i < 16.24k_d + 270 \\ ki > 370.17k_d - 21611 \\ k_i < 6561k_d + 15379 \end{cases}$$

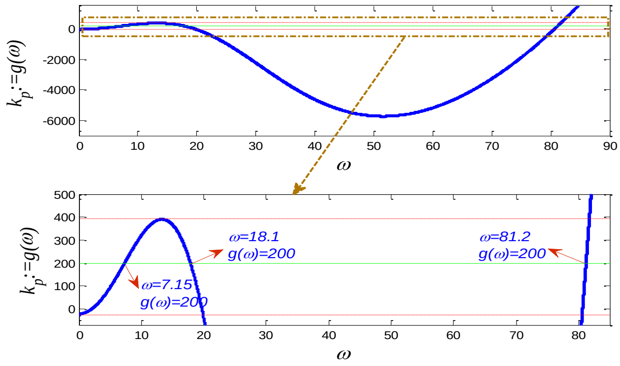

Fig. 5. The plot of function g(ω) with the line *kp*=200

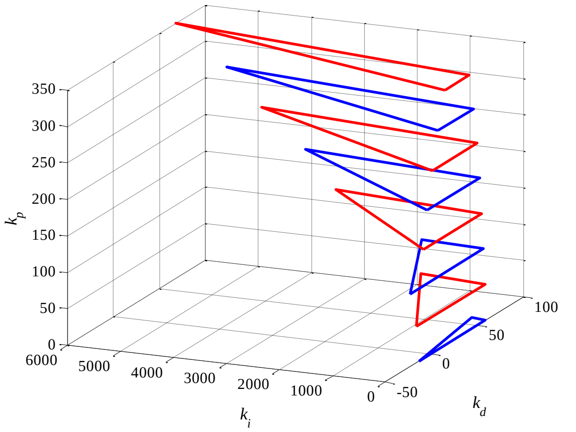

Fig. 6. The whole family of stabilizing PID controllers for nominal model of induction motor drive system

Therefore the family of robust PID controllers for the induction motor drive with uncertainties can be calculated by stabilizing the following two plants

$$\begin{cases} (1 + W(s)\Delta_{min})P(s) \\ (1 + W(s)\Delta_{max})P(s) \end{cases}.$$

The range of robust stabilizing PID parameters for induction motor faced to un-modeled dynamics is illustrated in Fig. 7-a. The admissible range of (k_i, k_d) for all admissible k_p is shown in Fig. 7-b.

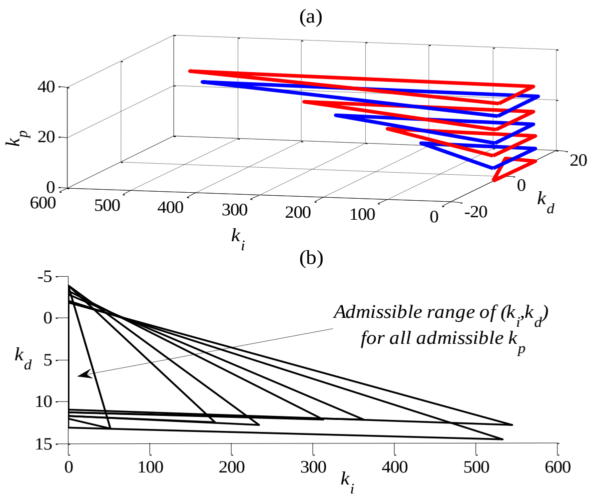

Fig. 7. (a) Robust stabilizing PID parameters for induction motor drive faced to un-modeled dynamics; (b) the admissible range of (k_i,k_d)

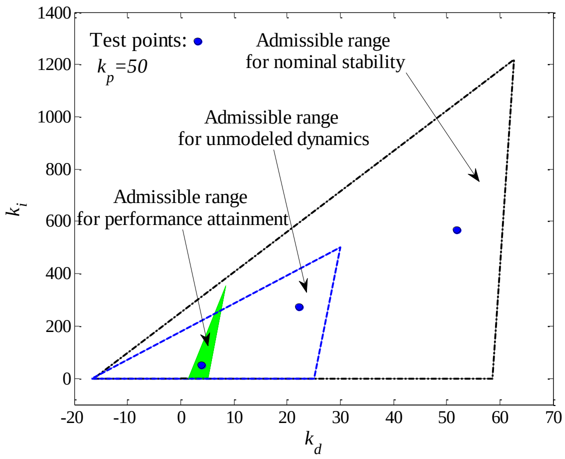

Fig. 8. The range of admissible (k_i,k_d)values that satisfy different control objectives for k_p=50

Now consider the problem of satisfying H_∞-norm on the complementary sensitivity function. The admissible range of(k_i, k_d)values that satisfy this performance criterion for $k_p = 50$ is shown in Fig. 8 together with regions corresponding to nominal stability and stability in the presence of un-modeled dynamics. The whole range of (k_i, k_d) values can be obtained similarly.

The step responses of closed loop system with controllers corresponding to test points marked in Fig. 8 are shown in Fig. 9. Applying the PID controller that satisfies performance consideration is resulted to decreasing the overshoot. Also, the control signals corresponding to selected controllers are plotted in Fig. 10. It can be seen that the control signals drift to zero after the reasonable times.

For better tracking of the proposed approach in this paper, the readers can refer to the many academic examples implemented in [15].

VI. CONCLUSION

In this paper, a robust control approach presented based on a general model for different types of electrical motor drives. The problem of robustness against un-modeled dynamics in motor drives is transformed to the problem of stabilizing a plant with an uncertain parameter. It is shown that knowing the frequency responses of motor drive system corresponding to maximum and minimum values of uncertainty is sufficient to calculate the family of robust stabilizing PID controllers. In fact, there is no need to plant mathematical model. Also it is illustrated that the problem of H_∞-norm achievement on the complementary sensitivity function can be solved by the same approach. Through the paper, it is assumed that the frequency response of plant is available. Such an assumption is often valid in many practical applications. Also, this is an assumption that has already been used several times in other papers dealing with controller synthesis using frequency domain data [5,7,11,13,15-17].