Sparse schrodinger operators

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SPARSE SCHRÖDINGER OPERATORS

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We study spectral properties of a family $(H_x^p)_{x\in X}$, indexed by a non-negative integer p, of one-dimensional discrete operators associated to an ergodic dynamical system $(T, X, \mathcal{B}, \mu)$ and defined for u in $\ell^2(\mathbb{Z})$ and n in $\mathbb{Z}$ by $(H_x^p u)(n) = u(n-p) + u(n+p) + V_x(n)u(n)$, where $V_x(n) = f(T^n x)$ and f is a real-valued measurable bounded map on X. In some particular cases, we prove that the nature of the spectrum does not change with p. Applications include some classes of random and quasi-periodic substitutional potentials.

1. Introduction

The one-dimensional discrete Schrödinger operator H, sometimes called the Jacobi matrix, is defined for u belonging to $\ell^2(\mathbb{Z})$ (the Hilbert space of square summable sequences), and for any integer n by

$$(Hu)(n) = u(n-1) + u(n+1) + V(n)u(n)\ , \tag{1.1}$$

where $(V(n))_{n\in\mathbb{Z}}$ is a bounded real *potential*. There has been a lot of interest for Schrödinger operators to be associated with a dynamical system $\mathcal{T} = (T, X, \mathcal{B}, \mu)$ as follows. For all x in X, let H_x be defined by

$$(H_x u)(n) = u(n-1) + u(n+1) + V_x(n)u(n)\ , \quad \forall n \in \mathbb{Z} \quad \text{and} \quad \forall u \in \ell^2(\mathbb{Z})\ , \tag{1.2}$$

where $V_x(n) = f(T^n x)$ and f is a real-valued bounded measurable function on X. Under the ergodicity of $\mathcal{T}$, the invariance of the spectral properties of such operators is true for μ-almost every operator H_x, which means that μ-almost all operators have the same spectrum and spectral components (see [1, 2, 4, 5, 6, 9, 13, 14, 18] for more details).

In this paper, we introduce more general operators defined on $\ell^2(\mathbb{Z})$, associated with the dynamical system $\mathcal{T}$ and indexed by a non-negative integer p. More precisely, we put for all x in X,

$$(H_x^p u)(n) = u(n-p) + u(n+p) + V_x(n)u(n)\ , \quad \forall n \in \mathbb{Z} \quad \text{and} \quad \forall u \in \ell^2(\mathbb{Z}) \tag{1.3}$$

where the potential V_x is defined as previously by a bounded mesurable map f from X to $\mathbb{R}$. Each H_x^p operator will be called a *p-sparse Schrödinger operator associated*

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with the dynamical system $\mathcal{T}$ and with the potential V_x. Our purpose is to study spectral properties of such operators, according to the values of p.

In Sec. 2, we first set up notations and terminologies of dynamical systems, spectral theory and random operators. We can also see that the notion of sparse Schrödinger operators is included in the more general theory of random ergodic operators described in details by A. Figotin and L. Pastur in [9]. According to [9] and under the ergodicity of $\mathcal{T}$, immediate spectral properties like invariance μ-almost everywhere of the spectrum and the spectral components, or absence μ-almost surely of the discrete component, can be deduced for sparse Schrödinger operators.

The specific form of p-sparse Schrödinger operators allows a more accurate understanding of their spectra and spectral components. In Part 3, the natural decomposition of $\ell^2(\mathbb{Z})$ in a direct sum of orthogonal subspaces, which are stable under each H_x^p, permits the study of p classical discrete one-dimensional Schrödinger operators associated to H_x^p, instead of studying H_x^p itself. We thus obtain p families of associated operators, each of them being defined for any x in X. In this case, we prove that each family is associated with the dynamical system $\mathcal{T}^p = (T^p, X, \mathcal{B}, \mu)$. Let us note that $\mathcal{T}^p$ is not necessarily ergodic.

When the dynamical system $\mathcal{T}$ is ergodic and minimal, we can cut X into m disjoint T^p-invariant closed subspaces, denoted by $X_0, \ldots, X_{m-1}$, where m is a non-negative integer depending on p and less than or equal to p. This result is due to W. H. Gottschalk and G. A. Hedlund (see [10]), and T. Kamae ([11]) for substitutional dynamical systems. We also refer the reader to [7]. Section 4 of this article states this theorem and discusses the spectral properties of each family of associated operators, according to the value of m. We prove in particular it suffices to restrict our attention to elements of X_0.

The forthcoming three sections concern special cases of potentials. In Part 5, we deal with the periodic case: $X = \mathbb{Z}/N\mathbb{Z}$, where N is a non-negative integer. For any x belonging to X, the sequences $(V_x(n))_{n\in\mathbb{Z}}$ are N-periodic. We prove that the spectrum Σ^p of any operator H_x^p is purely absolutely continuous and composed of N not necessarily disjoint bands (closed intervals of $\mathbb{R}$). This is exactly the same result as in the classical case of discrete unidimensional N-periodic Schrödinger operators, which can be found in [19, Chap. 4]. Moreover, we show that Σ^p can be explicitly described and depends only on p modulo N. More precisely, $\Sigma^p = \Sigma^{p \bmod(N)} = \Sigma^{N-(p \bmod(N))}$, for all $p \in \mathbb{N}^*$, and there exist exactly $[\frac{N}{2}] + 1$ possible spectra, which are Σ^1, Σ^2, ..., $\Sigma^{[\frac{N}{2}]}$ and Σ^N.

Section 6 treats the random case: we suppose that the $(V_x(n))_{n\in\mathbb{Z}}$ are independent identically distributed random variables. We prove that the general results stated for discrete one-dimensional Schrödinger operators with such potentials, can be extended to sparse Schrödinger operators. More precisely, we prove an analogue of the Kotani and Simon theorem: the absolutely component of μ-almost all operators is empty (see [13, 17], or [1], for more details). Moreover, when the density function is continuous on $\mathbb{R}$ and compactly supported, the spectrum Σ^p of μ-almost all operators H_x^p is pure point and equal to $[\alpha - 2, \beta + 2]$, if $[\alpha, \beta]$ is the support of

the density function. This is an analogue of the Kunz and Souillard theorem. In the same way we state an analogue of the Carmona, Klein and Martinelli theorem (random variables admitting a Bernoulli distribution). In these two extremal cases, we show that the nature of the spectrum does not change with p.

We analyze in Sec. 7 the case of substitutional potentials: the dynamical system $\mathcal{T}$ is generated by a primitive substitution ξ on a finite alphabet $\mathcal{A} = \{0, \ldots, r-1\}$. It is strictly ergodic, and we can apply the main theorem of Sec. 4. In the cases $m = 1$ or $m = p$, we state that the spectrum Σ^p of any operator H_x^p is the union of the spectra of any associated operator on X_0 and there is no absolutely continuous part almost everywhere. Moreover, when ξ has a constant length ℓ and $m = 1$ or p, the associated operators are classical Schrödinger operators with substitutional potentials. Finally, if $p = \ell^n$ for $n \geq 1$, then this new substitution is again ξ and the spectra of almost all associated operators have same nature which is the same for all Σ^{ℓ^n}.

2. Definitions and First Properties

2.1. *Dynamical systems and random operators*

Let $\mathcal{T} = (T, X, \mathcal{B}, \mu)$ be a dynamical system: X is an non-empty, compact metrizable space, $\mathcal{B}$ denotes the $\sigma-$algebra of Borel sets of X, μ is a probability measure on X and $T : X \to X$ is an automorphism (invertible transformation) of X, preserving the measure μ (that is to say for any $A \in \mathcal{B}$, $\mu(T^{-1}A) = \mu(A)$). The dynamical system $\mathcal{T}$ is said to be *ergodic* if each Borel subset A of X such that $T^{-1}A = A$ has a μ-measure equal to 0 or 1. It is called *uniquely ergodic* if there exists a unique T-invariant probability measure on X which turns out to be ergodic. If X has no closed T-invariant subspace other than $\emptyset$ and X itself, then $\mathcal{T}$ is a *minimal* dynamical system. When $\mathcal{T}$ is uniquely ergodic and minimal it is called *strictly ergodic*. In this article, we will always suppose $\mathcal{T}$ ergodic.

Let us denote by $(.,.)$ the inner product of $\ell^2(\mathbb{Z})$ and by $\|.\|_2$ its associated norm. Moreover, S denotes the shift operator on $\ell^2(\mathbb{Z})$.

We recall that a *random variable* on the probability space $(X, \mathcal{B}, \mu)$ is a real-valued $\mathcal{B}$-measurable function on X taking infinite values on a subset of X of μ-measure 1. A *random operator* A *on the probability space* $(X, \mathcal{B}, \mu)$ *of domain* $\ell^2(\mathbb{Z})$ is a map defined on X into the set of linear operators on $\ell^2(\mathbb{Z})$ by

$$A \ : \ x \longrightarrow A_x \ , \tag{2.1}$$

where A_x is for μ-almost every x in X a bounded linear operator on $\ell^2(\mathbb{Z})$, and such that for all u and v in $\ell^2(\mathbb{Z})$, the map $(Au, v) \ : \ x \in X \to (A_x u, v) \in \mathbb{R}$ is a random variable. If, in addition, μ-almost all operators A_x are self-adjoint, we say that A is *symmetric*. Moreover, if $\mathcal{T}$ is ergodic, and if there exists a homomorphism from the group $\{T^n \ ; \ n \in Z\}$ into a group $\{U_n \ ; n \in \mathbb{Z}\}$ of unitary operators on $\ell^2(\mathbb{Z})$ such that, for μ-almost all x in X,

$$A_{T^n x} = U_n A_x U_n^{-1} \ , \quad \forall n \in \mathbb{Z} \tag{2.2}$$

then A is called *ergodic* or *metrically transitive*. The best general reference for the random operators theory is [9].

2.2. *Spectral theory*

We recall that the *spectrum* $\sigma(H)$ of a self-adjoint continuous linear operator H is defined as the complement in $\mathbb{C}$ of the set of values λ for which $(H-\lambda Id)^{-1}$ exists and is a bounded linear operator on $\ell^2(\mathbb{Z})$. By self-adjointness and continuity of H, the set $\sigma(H)$ is a non-empty compact subset of $\mathbb{R}$. A real number λ for which there exist $u \in \ell^2(\mathbb{Z})$, $u \neq 0$, verifying $Hu = \lambda u$ is called an *eigenvalue* of H. The set of all eigenvalues of H is called the *point spectrum* $\sigma_p(H)$. The *pure point spectrum*, denoted by $\sigma_{pp}(H)$, is defined by

$$\sigma_{pp}(H) = \overline{\sigma_p(H)} \ ,$$

where $\overline{\sigma_p(H)}$ denotes the closure of the set $\sigma_p(H)$ in $\mathbb{R}$. The set $\sigma(H) \setminus \sigma_p(H)$ is the *continuous spectrum*. It can be cut into two parts, according to the Lebesgue decomposition of the spectral measure: the *absolutely continuous spectrum* $\sigma_{ac}(H)$, and the *singular continuous spectrum* $\sigma_{sc}(H)$. We thus have

$$\sigma(H) = \sigma_{pp}(H) \bigcup \sigma_{ac}(H) \bigcup \sigma_{sc}(H) \ , \tag{2.3}$$

and these sets are not necessarily disjoint. For more details, we refer the reader to Berthier ([3]) and to Dunford and Schwarz ([8]).

2.3. *Definition and immediate properties of sparse Schrödinger operators*

Let $(H_x^p)_{x\in X}$ be a family of p-sparse Schrödinger operators defined by Eq. (1.3) and associated with the ergodic dynamical system $\mathcal{T}$. We firstly note that each operator H_x^p can be written under the following form:

$$H_x^p = S^p + S^{-p} + V_x \tag{2.4}$$

where $(V_x u)(n) = V_x(n)u(n)$, for all u in $\ell^2(\mathbb{Z})$ and all n in $\mathbb{Z}$. Secondly, we remark that every operator H_x^p is a linear continuous self-adjoint operator from $\ell^2(\mathbb{Z})$ to itself. Its norm satisfies $\|H_x^p\| \leqq 2 + \sup_{n\in\mathbb{Z}} |V_x(n)| \leq 2 + \|(f)\|_\infty$.

By measurability of f, we can obtain, for given u and v in $\ell^2(\mathbb{Z})$, that the map $(H^p u, v)$ defined on X by $(H^p u, v)(x) = (H_x^p u, v)$ is a random variable. Hence H^p is a symmetric random operator on $(X, \mathcal{B}, \mu)$ in $\ell^2(\mathbb{Z})$. Moreover for all $x \in X$, H_x^p satisfies

$$H_{Tx}^p = S H_x^p S^{-1} \ , \tag{2.5}$$

and this leads us to Proposition 2.1.

Proposition 2.1. *The operator $H^p : x \to H_x^p$ for all x in X, is a symmetric ergodic random operator on $(X, \mathcal{B}, \mu)$ in $\ell^2(\mathbb{Z})$. The group of unitary operators on $\ell^2(\mathbb{Z})$ associated with the automorphism group $\{T^n ; n \in \mathbb{Z}\}$ is exactly $\{S^n ; n \in \mathbb{Z}\}$.*

Proof. We have already seen that H^p is a symmetric random operator. By Relation (2.5) and by induction, we deduce Eq. (2.2) with $U_n = S^n$ for all n. The ergodicity of the dynamical system $\mathcal{T}$ implies the metric transitivity of H^p. □

It is now possible to have information about the spectra and their components of operators H_x^p. Let us denote by $\sigma(H_x^p)$ (respectively $\sigma_{pp}(H_x^p)$, $\sigma_{ac}(H_x^p)$ and $\sigma_{sc}(H_x^p)$), the spectrum (resp. the pure point, absolutely continuous, and singular continuous parts of the spectrum), of each operator H_x^p. According to the previous part, we can remark that each $\sigma(H_x^p)$ is a non-empty compact subset of $\mathbb{R}$, included in the interval $[-2 - \sup_{n\in\mathbb{Z}} |V_x(n)|, 2 + \sup_{n\in\mathbb{Z}} |V_x(n)|]$.

Moreover, by Figotin and Pastur (see [9]), we obtain directly some properties of the spectrum of symmetric ergodic random operators.

Theorem 2.2. *Let H^p be as in Proposition* 2.1, *then*

(i) *there exists a non-empty compact set of $\mathbb{R}$, denoted by Σ^p, such that*

$$\Sigma^p = \sigma(H_x^p) \qquad \text{for } \mu\text{-almost all } x \in X \ ; \tag{2.6}$$

(ii) *there exist three closed subsets of $\mathbb{R}$, denoted respectively by Σ_{pp}^p, Σ_{ac}^p and Σ_{sc}^p, such that for μ-almost all x in X,*

$$\Sigma_{pp}^p = \sigma_{pp}(H_x^p) \ , \tag{2.7}$$

$$\Sigma_{ac}^p = \sigma_{ac}(H_x^p) \ , \tag{2.8}$$

$$\Sigma_{sc}^p = \sigma_{sc}(H_x^p) \ ; \tag{2.9}$$

(iii) *for μ-almost x in X, the spectrum of H_x^p admits no isolated eigenvalues of finite multiplicity, that is to say it is purely essential μ-almost everywhere:*

$$\sigma_{\text{dis}}(H_x^p) = \emptyset \quad \mu - a.e.; \tag{2.10}$$

(iv) *for any given λ in $\mathbb{R}$, $\mu(\{x \in X \ ; \ \lambda$ is an eigenvalue of finite multiplicity of $H_x^p\}) = 0$;*

(v) *for μ-almost all x in X and if $\sigma_{pp}(H_x^p)$ is not empty then it is locally uncountable.*

Proof. See Theorems 2.10, 2.11, 2.12 and 2.16 of [9]. □

Remark 2.1. When the dynamical system $\mathcal{T}$ is ergodic and minimal, it can be shown that

$$\Sigma^p = \sigma(H_x^p) \ , \quad \forall x \in X \ . \tag{2.11}$$

For details we again refer the reader to [9].

Thanks to random operator theory, we have obtained interesting results about nonrandomness of the spectrum and its components. But we do not know exactly neither their form nor the nature of the spectrum. It is the object of the next part of this article.

Remark 2.2. The random real-valued map $V(n)$ defined on X by $V(n)(x) = V_x(n)$, for all $x \in X$ where n is fixed, can be viewed as random variable on the probability space $(X, \mathcal{B}, \mu)$. Thus $(V(n))_{n\in\mathbb{Z}}$ is a sequence of random variables. Consequently p-sparse Schrödinger operators are special cases of random finite difference operators introduced by H. Kunz and B. Souillard in [14].

3. Decomposition of the Operator H_x^p

Let us denote by $(e_m)_{m\in\mathbb{Z}}$ the canonical orthonormal base of $\ell^2(\mathbb{Z})$: $e_m(n) = \delta_{n,m}$ for all n and m in $\mathbb{Z}$, where δ is the Kronecker symbol. For $i \in \{0, \ldots, p-1\}$, we consider the linear subspace of $\ell^2(\mathbb{Z})$ spanned by all vectors of the form e_{mp+i}, with $m \in \mathbb{Z}$. We denote it by $\mathcal{K}_i^p$ or $\mathcal{K}_i$ when there is no ambiguity. Thus

$$\mathcal{K}_i^p = \mathcal{K}_i = \text{ vect } \{e_{mp+i} \ , \ m \in \mathbb{Z}\} \ . \tag{3.1}$$

It is clear that every element v of $\mathcal{K}_i$ looks like $\ldots 0 \ldots 0 \ v_{i-p} \ 0 \ldots 0 \ v_i \ 0 \ldots 0 \ v_{i+p} 0 \ldots 0 \ v_{i+2p} \ldots$. Moreover, the $\mathcal{K}_i$ are mutually orthogonal, and $\ell^2(\mathbb{Z})$ is their orthogonal direct sum:

$$\ell^2(\mathbb{Z}) = \bigoplus_{i=0}^{p-1} \mathcal{K}_i \quad \text{with} \quad \mathcal{K}_i \perp \mathcal{K}_j \quad \text{for} \quad i \neq j \ . \tag{3.2}$$

Let us now consider the behavior of the operators H_x^p on each subspace $\mathcal{K}_i$.

Lemma 3.1. *For $0 \leqq i < p$ and if $\mathcal{K}_i$ is given by Relation (3.1), then $\mathcal{K}_i$ is stable under H_x^p.*

This lemma together with (3.2) implies Proposition 3.2.

Proposition 3.2. *For any x in X, the spectrum of H_x^p as well as its spectral components can be cut into p parts as follows:*

$$\sigma(H_x^p) = \bigcup_{i=0}^{p-1} \sigma(H_{x|\mathcal{K}_i}^p) \tag{3.3}$$

$$\sigma_\varepsilon(H_x^p) = \bigcup_{i=0}^{p-1} \sigma_\varepsilon(H_{x|\mathcal{K}_i}^p) \tag{3.4}$$

with $\varepsilon \in \{pp, ac, sc\}$.

Instead of studying H_x^p on $\ell^2(\mathbb{Z})$, we will do the study on each subspace $\mathcal{K}_i$. Before this, we have to look a little bit more at these subspaces.

3.1. *Study of subspaces $\mathcal{K}_i$*

First of all notice that every subspace $\mathcal{K}_i$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$. Indeed let us consider for each $0 \leq i < p$, the map $\phi_i^p = \phi_i : \mathcal{K}_i \to \ell^2(\mathbb{Z})$ defined for all $v \in \mathcal{K}_i$ and all n by

$$(\phi_i(v))(n) = v(np + i) \ .$$

The action of this map is illustrated by Fig. 1. Conversely, the map $\psi_i^p = \psi_i : \ell^2(\mathbb{Z}) \to \mathcal{K}_i$, given for any $u \in \ell^2(\mathbb{Z})$ and all m, by

$$(\psi_i(u))(m) = \begin{cases} u(n) & \text{if } m \equiv i \pmod p \\ 0, & \text{otherwise,} \end{cases}$$

is such that $\psi_i = (\phi_i)^{-1}$.

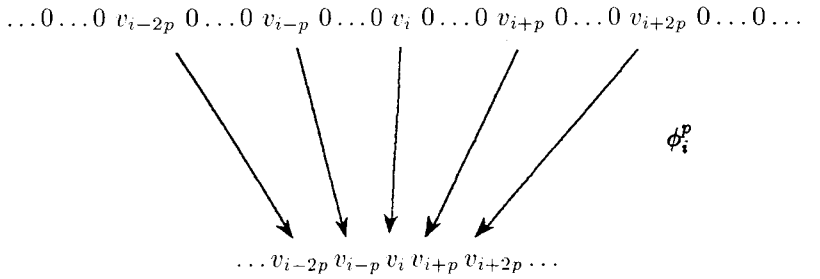

Fig. 1. The ϕ_i^p function

Moreover, ϕ_i and ψ_i, for $0 \leq i < p$, are linear continuous maps of norm equal to 1. Thus each $\mathcal{K}_i$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$. This also implies that every subspace $\mathcal{K}_i$ is isometrically isomorphic to any $\mathcal{K}_j$ $(0 \leqq i, j < p)$ and in particular the following lemma can be deduced.

Lemma 3.3. *For any $i \in \{0, \ldots, p-1\}$, if S denotes the shift operator on $\ell^2(\mathbb{Z})$, then*

$$S\mathcal{K}_0 = \mathcal{K}_{p-1} \tag{3.5}$$

$$S\mathcal{K}_i = \mathcal{K}_{i-1}\ , \quad \text{for} \quad 1 \leq i < p\ . \tag{3.6}$$

For $0 \leq i < p$, let us denote by $S_{|\mathcal{K}_i}$ the restriction of S to the subspace $\mathcal{K}_i$. Then

$$S_{|\mathcal{K}_0} : \mathcal{K}_0 \longrightarrow \mathcal{K}_{p-1}$$

$$S_{|\mathcal{K}_i} : \mathcal{K}_i \longrightarrow \mathcal{K}_{i-1}\ , \quad \text{for} \quad 1 \leq i < p\ .$$

Similarly, if $(S^{-1})_{|\mathcal{K}_i}$ is the restriction of S^{-1} on the subspace $\mathcal{K}_i$, we have

$$(S^{-1})_{|\mathcal{K}_i} : \mathcal{K}_i \longrightarrow \mathcal{K}_{i+1}\ , \quad \text{for} \quad 0 \leq i < p-1$$

$$(S^{-1})_{|\mathcal{K}_{p-1}} : \mathcal{K}_{p-1} \longrightarrow \mathcal{K}_0\ .$$

Finally it can be shown that

$$(S_{|\mathcal{K}_0})^{-1} = (S^{-1})_{|\mathcal{K}_{p-1}} \tag{3.7}$$

$$(S_{|\mathcal{K}_i})^{-1} = (S^{-1})_{|\mathcal{K}_{i-1}} \quad \text{, for} \quad 1 \leq i < p\ . \tag{3.8}$$

Relations (3.5) to (3.8) lead to a dependence of the ϕ_i functions.

Proposition 3.4. *For any $i \in \{0, \ldots, p-2\}$, we have*

$$\phi_i \circ S_{|\mathcal{K}_{i+1}} = \phi_{i+1} \ . \tag{3.9}$$

Moreover if $i = p-1$,

$$\phi_{p-1} \circ S_{|\mathcal{K}_0} = S \circ \phi_0 \ . \tag{3.10}$$

Proof. The proof is left to the reader. □

Remark 3.1. This proposition implies that, for any i in $\{0, \ldots, p-2\}$ (and respectively for $i = p-1$), the following diagrams commute:

$$\begin{array}{ccc} \mathcal{K}_{i+1} & \xrightarrow{S_{|\mathcal{K}_{i+1}}} & \mathcal{K}_i \\ {\scriptstyle \phi_{i+1}}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ \ell^2(\mathbb{Z}) & \xrightarrow[Id]{} & \ell^2(\mathbb{Z}) \end{array} \qquad \begin{array}{ccc} \mathcal{K}_0 & \xrightarrow{S_{|\mathcal{K}_0}} & \mathcal{K}_{p-1} \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi_{p-1}} \\ \ell^2(\mathbb{Z}) & \xrightarrow[S]{} & \ell^2(\mathbb{Z}) \end{array}$$

Proposition 3.5. *For $0 \leqq i < p$, the subspace $\mathcal{K}_i$ is stable under S^p, and if $(S^p)_{|\mathcal{K}_i}$ denotes the restriction of S^p to $\mathcal{K}_i$, then*

$$S \circ \phi_i = \phi_i \circ (S^p)_{|\mathcal{K}_i} \ . \tag{3.11}$$

This signifies that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_i & \xrightarrow{S^p_{|\mathcal{K}_{i+1}}} & \mathcal{K}_i \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_i} \\ \ell^2(\mathbb{Z}) & \xrightarrow[S]{} & \ell^2(\mathbb{Z}) \end{array}$$

Proof. The proof is left to the reader. □

3.2. *Associated operators*

We have already noticed that we had to study each operator $H^p_{x|\mathcal{K}_i}$, for $0 \leq i < p$. Because each subspace $\mathcal{K}_i$ is isometrically isomorphic to $\ell^2(\mathbb{Z})$, we will lift this study from $\mathcal{K}_i$ to $\ell^2(\mathbb{Z})$ by putting for all $i \in \{0, \ldots, p-1\}$ and all $x \in X$:

$$\tilde{H}^{p,i}_x = \phi_i \circ H^p_{x|\mathcal{K}_i} \circ (\phi^p_i)^{-1} \ . \tag{3.12}$$

The $\tilde{H}^{p,i}_x$ are continuous linear self-adjoint operators on $\ell^2(\mathbb{Z})$. They are the H^p_x *associated operators*. Each operator $\tilde{H}^{p,i}_x$ being unitarily equivalent to $H^p_{x|\mathcal{K}_i}$, the following proposition can be deduced.

Proposition 3.6. *For all $i \in \{0, \ldots, p-1\}$ and all $x \in X$,*

$$\sigma(\tilde{H}^{p,i}_x) = \sigma(H^p_{x|\mathcal{K}_i}) \tag{3.13}$$

$$\sigma_\varepsilon(\tilde{H}^{p,i}_x) = \sigma_\varepsilon(H^p_{x|\mathcal{K}_i}) \ , \quad \text{with} \quad \varepsilon \in \{pp, ac, sc\} \ . \tag{3.14}$$

Consequently we can conclude by Theorem 3.7 concerning the spectrum of H_x^p.

Theorem 3.7. *Under the ergodicity of the dynamical system $\mathcal{T}$, and with notations of Theorem 2.2, we have*

$$\Sigma^p = \bigcup_{i=0}^{p-1} \sigma(\tilde{H}_x^{p,i}) \tag{3.15}$$

$$\Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \sigma_\varepsilon(\tilde{H}_x^{p,i}) \tag{3.16}$$

for $\varepsilon \in \{pp, ac, sc\}$ and μ-almost all x in X.

Proof. It is an immediate consequence of Theorem 2.2, together with Propositions 3.2 and 3.6. □

Remark 3.2. It is an interesting fact that the $\tilde{H}_x^{p,i}$ spectra are "globally" invariants, that is to say their union is invariant μ-almost everywhere. But we do not know anything about the behavior of each one of these operators.

Remark 3.3. If moreover the dynamical system $\mathcal{T}$ is minimal, then Σ^p is equal to the union of the spectra of $\tilde{H}_x^{p,i}$ for any x in X.

3.3. *Immediate properties of associated operators*

In this section, we see the strong links between the operators $\tilde{H}_x^{p,i}$. We begin by a new characterization of them.

Proposition 3.8. *For $0 \leq i < p$ and all $x \in X$, $\tilde{H}_x^{p,i}$ is a linear operator defined on $\ell^2(\mathbb{Z})$ by*

$$\tilde{H}_x^{p,i} = S + S^{-1} + \tilde{V}_x^{p,i} , \tag{3.17}$$

where

$$\tilde{V}_x^{p,i}(n) = V_x(np + i) , \quad \forall n \in \mathbb{Z} . \tag{3.18}$$

Proof. The result directly follows from the definition of $\tilde{H}_x^{p,i}$. □

This proposition means that the sequence of potentials $\tilde{V}_x^{p,i}$ is a subsequence of V_x.

Proposition 3.9. *For $0 \leq i < p$ and all $x \in X$, we have the following relations:*

$$\tilde{H}_{Tx}^{p,p-1} = S\tilde{H}_x^{p,0}S^{-1} \tag{3.19}$$

$$\tilde{H}_{Tx}^{p,i} = \tilde{H}_x^{p,i+1}. \tag{3.20}$$

Proof. By definition of operators $\tilde{H}_x^{p,i}$, and according to Propositions 3.4 and 3.5, we complete the proof. □

Corollary 3.10. *If $i \in \{0, \ldots, p-1\}$, and if j is a non-negative integer such that $i + j = np + m$ where $0 \leq m < p$ and $n \in \mathbb{N}$, then*

$$\tilde{H}_{T^j x}^{p,i} = S^n \tilde{H}_x^{p,m} S^{-n} . \tag{3.21}$$

Proof. The result follows inductively from the previous proposition. □

Remark 3.4. In particular, for all $x \in X$ and all $i \in \{0, \ldots, p-1\}$, we deduce that $\tilde{H}_x^{p,i} = \tilde{H}_{T_x^i}^{p,0}$.

Remark 3.5. Corollary 3.10 implies for example that $\tilde{H}_{Tx}^{p,i}$ and $\tilde{H}_x^{p,i+1}$ are unitarily equivalent. They also have same spectra and same spectral components.

Corollary 3.11. *Let i be given in $\{0, \ldots, p-1\}$. For all $x \in X$, $\tilde{H}_x^{p,i}$ verifies*

$$\tilde{H}_{T^p x}^{p,i} = S \tilde{H}_x^{p,i} S^{-1} \ . \tag{3.22}$$

Proof. Since $0 \leq i < p$, Corollary 3.11 can be applied with $n = 1$ and $m = i$. Thus the proof is complete. □

To end this part, we put $g_i = f \circ T^i$, for $0 \leq i < p$. Then g_i is a bounded measurable function from X to $\mathbb{R}$, and

$$\tilde{V}_x^{p,i}(n) = g_i\big((T^p)^n x\big) \ , \quad \forall n \in \mathbb{Z} \ , \quad \forall x \in X \ . \tag{3.23}$$

This permits us to conclude with a theorem.

Theorem 3.12. *Let i be given in $\{0, \ldots, p-1\}$. Then $(\tilde{H}_x^{p,i})_{x \in X}$ is a family of discrete unidimensional Schrödinger operators associated with the dynamical system $(T^p, X, \mathcal{B}, \mu)$.*

Proof. According to Propositions 3.8 and 3.11 with Relation (3.23), we complete the proof. □

Remark 3.6. We can also say that $\tilde{H}^{p,i}$ is a symmetric random operator on the probability space $(X, \mathcal{B}, \mu)$ of domain $\ell^2(\mathbb{Z})$. But we do not know if it is ergodic or not: in the general case, the dynamical system $(T^p, X, \mathcal{B}, \mu)$ is not supposed to be ergodic! This is the object of the following section.

4. The Decomposition of a Dynamical System and Its Applications

Let $\mathcal{T}$ be an ergodic dynamical system. If p is a non-negative integer, we denote by $\mathcal{T}^p$ the new dynamical system $(T^p, X, \mathcal{B}, \mu)$. In this section, we are concerning into the ergodicity of the dynamical system $\mathcal{T}^p$. In general case, we are not able to give any answer; but when $\mathcal{T}$ is ergodic and minimal, the following theorem is a useful tool which is given in [7] (see also [11, 10]).

Theorem 4.1. (GOTTSCHALK, HEDLUND, KAMAE). *Let $\mathcal{T} = (T, X, \mathcal{B}, \mu)$ be a minimal and ergodic dynamical system. Let p be a non-negative integer. Then there exists a finite partition of X, denoted by $\{X_0, \ldots, X_{m-1}\}$, such that*

(i) $\bigcup_{k=0}^{m-1} X_k = X$ *and* $X_k \cap X_\ell = \emptyset$ *if* $k \neq \ell$;
(ii) *each X_k is a closed non-empty subset of X;*
(iii) *each X_k is T^p-invariant: $T^p X_k = X_k$;*
(iv) *X_k does not admit any closed T^p-invariant proper subspace;*
(v) *the partition is cyclic: $TX_0 = X_1, \ldots, TX_{m-2} = X_{m-1}, TX_{m-1} = X_0$.*

This partition is unique up to a cyclic permutation of its terms.

We illustrate this result by Fig. 2.

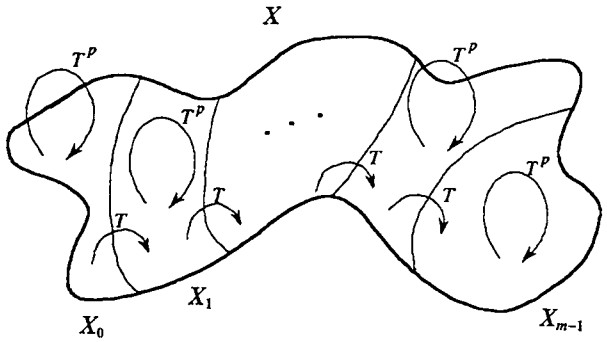

Fig. 2. The decomposition of a dynamical system

Remark 4.1. Under the notations of Theorem 4.1, each dynamical system $(T^p_{|X_k}, X_k, \mathcal{B}_{|X_k}, \mu_k)$ is minimal, where $\mu_k(A) = \mu(A \cap X_k)/\mu(X_k)$ for any Borel subset A of X. The partition is also said to be *minimal*. Moreover, $\mu(X_k) = 1/m$.

Remark 4.2. The non-negative integer m defined in Theorem 4.1 depends on p and is less than or equal to it. From now on will we denote $m = \delta(p)$ and $\delta(.)$ is called the *decomposition function* of powers of T. It is linked to p by $\delta(p) \mid p$.

In all this section, we will suppose $\mathcal{T}$ ergodic and minimal. We also state some corollaries, which can be found in [7].

Corollary 4.2. *We suppose that the dynamical system $\mathcal{T}$ is strictly ergodic. Then, under notations of Theorem 4.1, each dynamical system $(T^{\delta(p)}_{|X_k}, X_k, \mathcal{B}_{|X_k}, \mu_k)$ is uniquely ergodic.*

Moreover, Remark 4.1 and Corollary 4.2 imply the following result.

Corollary 4.3. *We suppose that the dynamical system $\mathcal{T}$ is strictly ergodic. If, moreover, $\delta(p) = p$, then each dynamical system $(T^p_{|X_k}, X_k, \mathcal{B}_{|X_k}, \mu_k)$ is strictly ergodic.*

The following theorem yields information about spectral behavior of the associated operators.

Proposition 4.4. *If Σ^p is the spectrum of H^p_x for all x in X, then*

$$\Sigma^p = \bigcup_{i=0}^{p-1} \sigma(\tilde{H}^{p,i}_x) \ , \quad \forall x \in X_0 \ . \tag{4.1}$$

Moreover, if Σ^p_{pp}, Σ^p_{ac} and Σ^p_{sc} are the spectral components of the spectrum of μ-almost all H^p_x,

$$\Sigma^p_\varepsilon = \bigcup_{i=0}^{p-1} \sigma_\varepsilon(\tilde{H}^{p,i}_x) \tag{4.2}$$

for $\varepsilon \in \{pp, ac, sc\}$ and for μ_0-almost all x in X_0.

Remark 4.3. In our study, we have to consider each associated operator $\tilde{H}_x^{p,i}$ on the dynamical system $\mathcal{T}^p$. Theorem 4.1 induces us to restrict our attention to $\tilde{H}_x^{p,i}$ on each "sub"-dynamical system $\mathcal{T}_k^p = (T_{|X_k}^p, X_k, \mathcal{B}_{|X_k}, \mu_k)$. Now Proposition 4.4 permits us to study the associated operators only on the dynamical system $\mathcal{T}_0^p = (T_{|X_0}^p, X_0, \mathcal{B}_{|X_0}, \mu_0)$.

Proof. Let i be given in $\{0, \dots, p-1\}$ and k be in $\{0, \dots, \delta(p)-1\}$. By minimality and ergodicity of $\mathcal{T}$, we know that Σ^p is the spectrum of any operator H_x^p, and (4.1) is deduced.

By Theorem 4.1, $X_k = T^k X_0$ and if y is given in X_k, then there exists $x \in X_0$ such that $y = T^k x$. This implies $\tilde{H}_y^{p,i} = \tilde{H}_{T^k x}^{p,i}$. If $i + k = np + m$ where $n \geq 0$ and $0 \leq m < p$, then by Corollary 3.10, $\tilde{H}_y^{p,i} = S^n \tilde{H}_x^{p,m} S^{-n}$, and for $\varepsilon \in \{pp, ac, sc\}$,

$$\bigcup_{i=0}^{p-1} \sigma_\varepsilon(\tilde{H}_y^{p,i}) = \bigcup_{m=0}^{p-1} \sigma_\varepsilon(\tilde{H}_x^{p,m}) \ .$$

This last equality is verified for any $y \in X_k$.

If there exist $A_0 \subset X_0$ of μ_0-measure 1 on which the spectral component Σ_ε^p is not equal to the union of $\sigma_\varepsilon(\tilde{H}_x^{p,i})$, then for any $1 \leq k < \delta(p)$, $A_k = T^k A_0 \subset X_k$ has the same property as A_0. We put $A = \cup_{k=0}^{\delta(p)-1} A_k \subset X$. The set A verifies $\mu(A) = 1$ and has the same property as A_0. This is a contradiction with Theorem 2.2 and the proof is complete. □

Remark 4.4. We have an analogue of Proposition 4.4 in replacing $\mathcal{T}_0^p$ by any $\mathcal{T}_k^p$.

As a direct consequence of Remark 3.4 and Proposition 4.4, we can state Corollary 4.5.

Corollary 4.5.

$$\Sigma^p = \bigcup_{i=0}^{p-1} \sigma(\tilde{H}_{T^i x}^{p,0}) \ , \quad \forall\, x \in X_0 \tag{4.3}$$

$$\Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \sigma_\varepsilon(\tilde{H}_{T^i x}^{p,0}) \ , \quad \forall\, \varepsilon \in \{pp, ac, sc\} \text{ and } \mu_0 - a.e. \tag{4.4}$$

Let us now consider associated operators $\tilde{H}_x^{p,i}$ on the dynamical system $\mathcal{T}_0^p$. For any i belonging to $\{0, \dots, p-1\}$, $(\tilde{H}_x^{p,i})_{x \in X_0}$ is a family of Schrödinger operators associated with $\mathcal{T}_0^p$. But we do not know whether $\mathcal{T}_0^p$ is ergodic in general case: in fact, when $\mathcal{T}$ is strictly ergodic, it depends on the values taking by $\delta(p)$.

Proposition 4.6. *Let $\mathcal{T}$ be strictly ergodic dynamical system and p a non-negative integer such that $\delta(p) = p$.*

(i) *There exist p non-empty compact subsets of $\mathbb{R}$, denoted by $\tilde{\Sigma}^{p,0}, \dots, \tilde{\Sigma}^{p,p-1}$ verifying*

$$\Sigma^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}^{p,i} \ . \tag{4.5}$$

Each of them is the spectrum of operators $\tilde{H}_x^{p,i}$ $(\forall x \in X_0)$.

(ii) *For* $\varepsilon \in \{pp, ac, sc\}$ *there exists a compact subset* $\tilde{\Sigma}_\varepsilon^{p,i}$ *of* $\mathbb{R}$ *such that* $\sigma_\varepsilon(\tilde{H}_x^{p,i}) = \tilde{\Sigma}_\varepsilon^{p,i}$ *for* μ_0*-almost all* x *in* X_0. *Moreover*

$$\Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}_\varepsilon^{p,i} . \tag{4.6}$$

Proof. By Corollary 4.3, Proposition 4.4, Theorem 2.2 and Corollary 2.3. □

Remark 4.5. Under notations of Proposition 4.6, Remark 3.4 implies that the spectrum $\tilde{\Sigma}^{p,i}$ of any $\tilde{H}_x^{p,i}$ on X_0 is also the spectrum of $\tilde{H}_y^{p,0}$, for any $y \in X_i$.

We now will describe explicit cases.

5. The Periodic Case

Let us consider $X = \mathbb{Z}/N\mathbb{Z}$, where $N \in \mathbb{N}^*$. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of X and by μ the counting measure on X (defined for all $0 \leq j < N$ by $\mu(\{j\}) = \frac{1}{N}$). The transformation $T : x \to x + 1$ is an invertible measure-preserving transformation of X. The dynamical system $\mathcal{T} = (T, X, \mathcal{B}, \mu)$ is strictly ergodic. If f is a measurable bounded map from X to $\mathbb{R}$, we define the potential for all $x \in X$ by

$$V_x(n) = f(T^n x) , \quad \forall n \in \mathbb{Z} . \tag{5.11}$$

In addition, we suppose f such that the sequence $(V_x(n))_n$ is exactly N-periodic. Let p be given in $\mathbb{N}^*$. In this section, we study p-sparse Schrödinger operators associated with the dynamical system $\mathcal{T}$ and with the potential (5.1). For general results on discrete one-dimensional periodic Schrödinger operators, we refer the reader to Toda [19].

5.1. *First properties of the associated operators*

We can now introduce the associated operators $\tilde{H}_x^{p,i}$ defined by Eqs. (3.12) and (3.17)–(3.18). For all $i \in \{0, \ldots, p-1\}$ and all $x \in X$, $\tilde{H}_x^{p,i}$ is given by $\tilde{H}_x^{p,i} = S + S^{-1} + \tilde{V}_x^{p,i}$, where

$$\tilde{V}_x^{p,i}(n) = V_x(np + i) = f(x + np + i) \quad \forall n \in \mathbb{Z} . \tag{5.2}$$

Proposition 5.1. *Let* i *be given in* $\{0, \ldots, p-1\}$ *and* $x \in X$. *Then* $\tilde{H}_x^{p,i}$ *is a discrete uni-dimensional periodic Schrödinger operator, where the period* $\tilde{N}_p$ *does not depend on* i *and is given by*

$$\tilde{N}_p = \frac{N}{\gcd(p, N)} . \tag{5.3}$$

Proof. It is easy to see that $\tilde{H}_x^{p,i}$ is a periodic Schrödinger operator. For the calculation of its period, which clearly does not depend on i, we consider two cases: $p = \alpha N$ or not.

First, if $p = \alpha N, \alpha \geq 1$, $\tilde{V}_x^{p,i}(n) = f(x+i)$ is constant for all n. Thus $\tilde{N}_p = 1$. Remark that $\gcd(N,p) = N$. Suppose now $p \neq \alpha N$. If $1 \leq p < N$, the period $\tilde{N}_p$ is such that $\tilde{N}_p \times p$ is the lowest multiple of N, which is of course a multiple of p. Therefore

$$\tilde{N}_p = \frac{\operatorname{lcm}(N,p)}{p} .$$

If $p > N$, let us consider the Euclidian division of p by N: $p = qN + r$, where $0 < r < N$ ($p \neq \alpha N$). It appears that $\tilde{V}_x^{p,i}(n) = f(x+nr+i)$ so that $\tilde{N}_p = \operatorname{lcm}(N,r)/r$.

Moreover for all a and b non-negative integers, $ab = \operatorname{lcm}(a,b) \times \gcd(a,b)$. Thus

$$\frac{\operatorname{lcm}(a,b)}{b} = \frac{a}{\gcd(a,b)} .$$

Finally, notice that for all $\alpha \geq 1$, $\gcd(a,b) = \gcd((a+\alpha b),b)$, and $\gcd(N,r) = \gcd(N,p)$. This concludes the proof. □

Recall that the spectrum of a discrete one-dimensional P-periodic Schrödinger operator is well known. Namely it is purely absolutely continuous and composed of P *bands*, which are closed intervals of $\mathbb{R}$. These bands are not necessarily disjoint. For a treatment of this case, we refer the reader to [19, Chap. 4]. We can now formulate a similar result for periodic sparse Schrödinger operators.

Theorem 5.2. *For all $x \in X$, Σ^p is absolutely continuous and defined by*

$$\Sigma^p = \bigcup_{i=0}^{p-1} \sigma(\tilde{H}_x^{p,i}) . \tag{5.4}$$

There are at most $p\tilde{N}_p$ bands in the spectrum.

Proof. By minimality and ergodicity of $\mathcal{T}$, the spectrum is the same for all x (Corollary 2.3). From the previous proposition, the associated operators are $\tilde{N}_p$-periodic. Thus, the spectrum of each associated operator $\tilde{H}_x^{p,i}$ is purely absolutely continuous and composed of $\tilde{N}_p$ bands. Theorem 3.7 completes the proof. □

5.2. *Decomposition function and consequences on the spectrum*

By the strict ergodicity of the dynamical system $\mathcal{T}$, we can apply Theorem 4.1. Let $\delta(p)$ be the decomposition function corresponding to p. Thanks to the peculiar form of X and to the definition of T, we have information about $\delta(p)$.

Theorem 5.3. *If $X = \mathbb{Z}/N\mathbb{Z}$ and $Tx = x+1$ on X, then the decomposition function is given for any non-negative integer p, by*

$$\delta(p) = \gcd(p,N) . \tag{5.5}$$

Moreover, each dynamical system $(T^p_{|X_k}, X_k, \mathcal{B}_{|X_k}, \mu_k)$ is strictly ergodic.

Proof. It is clear that $\delta(p)$ is linked with the period of T^p. By Relation (5.3), T^p is $\tilde{N}_p$-periodic with $\tilde{N}_p = N/\gcd(N,p)$. This means that each T^p-invariant subset

of X admits exactly $\tilde{N}_p$ elements. By definition of the decomposition function, there are $\delta(p)$ such subsets. Thus $\delta(p) \times \tilde{N}_p = N$, and we deduce the expression of $\delta(p)$. The second statement is clear. □

Remark 5.1. This theorem implies that $1 \leq \delta(p) \leq \min(p, N)$.

Remark 5.2. When p and N are relatively prime we know immediately that $\delta(p) = 1$. Moreover

$$\delta(p) = p \Longleftrightarrow N = \alpha p \quad \text{where} \quad \alpha \geq 1 \,,$$
$$\delta(p) = N \iff p = \beta N \quad \text{where} \quad \beta \geq 1 \,.$$

Remark 5.3. We suppose X_0 to be the subset of the partition $\{X_0, \ldots, X_{\delta(p)-1}\}$ containing 0. This theorem allows us to an analogue of Proposition 4.6, whatever the value of $\delta(p)$.

Theorem 5.4. *Let p be a non-negative integer, and $(H_x^p)_{x\in X}$ be a family of p-sparse N-periodic Schrödinger operators. Then*

$$\Sigma^p = \bigcup_{i=0}^{\delta(p)-1} \tilde{\Sigma}^{p,i} \,, \tag{5.6}$$

where $\tilde{\Sigma}^{p,i}$ is the spectrum of every operator $\tilde{H}_x^{p,i}$ when $x \in X_0$. In the same way, for $\varepsilon \in \{pp, ac, sc\}$, the ε-spectral component of the spectrum Σ^p is given by

$$\Sigma_\varepsilon^p = \bigcup_{i=0}^{\delta(p)-1} \tilde{\Sigma}_\varepsilon^{p,i} \,, \tag{5.7}$$

with $\sigma_\varepsilon(\tilde{H}_x^{p,i}) = \tilde{\Sigma}_\varepsilon^{p,i}$ for μ_0-almost all x in X_0.

Proof. Thanks to strict ergodicity given by Theorem 5.3, and as in the proof of Proposition 4.6, we see that

$$\Sigma^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}^{p,i} \quad \text{and} \quad \Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}_\varepsilon^{p,i} \,.$$

But the main property of $\delta(p)$ is $\delta(p) \mid p$. Let us consider α the non-negative integer verifying $p = \alpha\delta(p)$. Thus

$$\Sigma^p = \bigcup_{j=0}^{\alpha-1} \left(\bigcup_{i=0}^{\delta(p)-1} \tilde{\Sigma}^{p,i+j\delta(p)} \right) \,.$$

For any $j \in \{0, \ldots, \alpha-1\}$ and $i \in \{0, \ldots, \delta(p)-1\}$ we know that $\tilde{\Sigma}^{p,i} = \sigma(\tilde{H}_x^{p,i})$ for all $x \in X_0$. By Proposition 3.9 and for any x in X_0, we see that $\tilde{H}_x^{p,i+j\delta(p)} = \tilde{H}_{(T^{\delta(p)})^j x}^{p,i}$ and $T^{\delta(p)}x \in X_0$. Thus $\tilde{\Sigma}^{p,i} = \sigma(\tilde{H}_{(T^{\delta(p)})^j x}^{p,i}) = \tilde{\Sigma}^{p,i+j\delta(p)}$, and Relation (5.6) is proved. For (5.7) equalities hold μ_0-almost everywhere. □

Corollary 5.5. *When $p = \beta N$ where $\beta \geq 1$, then we know exactly Σ^p.*

$$\Sigma^p = \bigcup_{i=0}^{N-1} [f(i) - 2, f(i) + 2] \ . \tag{5.8}$$

Proof. By Remark 5.2, $\delta(p) = N$. Thus $X_0 = \{0\}$ and Theorem 5.4 implies for $\varepsilon \in \{pp, ac, sc\}$ that

$$\Sigma^p = \bigcup_{i=0}^{N-1} \sigma(\tilde{H}_0^{p,i}) \quad \text{and} \quad \Sigma_\varepsilon^p = \bigcup_{i=0}^{N-1} \sigma_\varepsilon(\tilde{H}_0^{p,i}) \ .$$

But, as we have already noticed in the proof of Proposition 5.1, $\tilde{V}_0^{p,i}$ is a constant sequence equal to $f(i)$. In this case, we know (*cf* [19]), that $\sigma(\tilde{H}_0^{p,i}) = [f(i) - 2, f(i) + 2]$. This concludes the proof. □

The following corollary is another direct consequence of Proposition 5.4.

Corollary 5.6. *If p is a non-negative integer and if $q = \alpha N + p$ when $\alpha \geq 1$, then*

$$\Sigma^p = \Sigma^q \ . \tag{5.9}$$

Proof. By Theorem 5.4, Σ^p (respectively Σ^q) depends on the sets $\tilde{\Sigma}^{p,0}, \ldots, \tilde{\Sigma}^{p,\delta(p)-1}$ (respectively on the sets $\tilde{\Sigma}^{q,0}, \ldots, \tilde{\Sigma}^{q,\delta(q)-1}$), which are the spectra of $\tilde{H}_x^{p,0}, \ldots, \tilde{H}_x^{p,\delta(p)-1}$ on X_0^p (respectively of $\tilde{H}_y^{q,0}, \ldots, \tilde{H}_y^{q,\delta(q)-1}$ on X_0^q). But, as we have already noticed in the proof of Proposition 5.1, $\delta(q) = \delta(\alpha N + p) = \delta(p)$ and therefore $\tilde{N}_p = \tilde{N}_q$.

Moreover X_0^p (respectively X_0^q), contains 0 and is exactly equal to the set $\{0, T^p 0, \ldots, T^{(\tilde{N}_p - 1)p} 0\}$ (respectively $X_0^q = \{0, T^q 0, \ldots, T^{(\tilde{N}_q - 1)q} 0\}$). For $0 \leq n < \tilde{N}_p$, $nq \bmod (N) \equiv np \bmod (N)$ and then $T^{nq}0 = T^{np}0$. This implies that $X_0^p = X_0^q$. Let us now compare $\tilde{\Sigma}^{p,i}$ and $\tilde{\Sigma}^{q,i}$ for $i = 0, \ldots, \delta(p) - 1$. For all n and all $x \in X_0$, let us note that $x + nq + i \bmod (N) \equiv x + np + i \bmod (N)$ and $\tilde{V}_x^{q,i}(n) = \tilde{V}_x^{p,i}(n)$. Thus $\tilde{H}_x^{q,i} = \tilde{H}_x^{p,i}$ and the proof is complete. □

Remark 5.4. This corollary means that the spectrum does not change if p is replaced by $p + \alpha N$, with $\alpha \geq 1$. Thus the spectrum of H_x^p with $p = \alpha N$ is exactly Σ^N and we find again the result of Corollary 5.5.

Remark 5.5. Now, Σ^p admits a most $\delta(p)\tilde{N}_p$ bands, that is to say at most N bands.

Proposition 5.7. *Let p be given in $\{1, \ldots, N-1\}$. We put $q = N - p$. Then*

$$\Sigma^p = \Sigma^q \ . \tag{5.10}$$

Proof. When N is even and $p = N/2$, the result is evident. We suppose now $1 \leq q < p < N$. It is easy to prove that $\delta(p) = \delta(q)$. Consequently $\tilde{N}_p = \tilde{N}_q$. Thus the associated operators $\tilde{H}_x^{p,i}$ and $\tilde{H}_x^{q,i}$ are periodic Schrödinger operators with same periods, and according to Theorem 5.4 their spectra are decomposed into the same

number of parts: $\Sigma^p = \bigcup_{i=0}^{\delta(p)-1} \sigma(\tilde{H}_x^{p,i})$ for all x in X_0^p, and $\Sigma^q = \bigcup_{i=0}^{\delta(p)-1} \sigma(\tilde{H}_x^{q,i})$ for all x in X_0^q.

But $X_0^p = \{0, T^p 0, \ldots, T^{(\tilde{N}_p-1)p}0\}$ and $X_0^q = \{0, T^q 0, \ldots, T^{(\tilde{N}_p-1)q}0\}$. For $1 \leq n < \tilde{N}_p$, it is clear that $(\tilde{N}_p - n)p \bmod (N) \equiv N(p/\gcd(N,p)) - np \bmod (N) \equiv n(N-p) \bmod (N)$, and $T^{(\tilde{N}_p-n)p}0 = T^{nq}0$. Hence $X_0^p = X_0^q$.

Moreover for $0 \leq i < \delta(p)$, all x in X_0 and all n, $x + n(N-q) + i \bmod (N) \equiv x + (-n)q + i \bmod (N)$ and $\tilde{V}_x^{p,i}(n) = \tilde{V}_x^{q,i}(-n)$. We use the Floquet theorem which characterizes elements of the spectrum of a P-periodic one-dimensional discrete Schrödinger operator H as follows (for more details see [19]).

$$\lambda \in \sigma(H) \Longleftrightarrow \exists u \in \ell^\infty(\mathbb{Z}) \text{ such that} \begin{cases} Hu = \lambda u \\ u(n+P) = \rho u(n) \ \forall n \ , \\ \text{where } \rho \in \mathbb{C} \text{ and } |\rho| = 1. \end{cases}$$

For any $u \in \ell^\infty(\mathbb{Z})$, we put $\bar{u}(n) = u(-n)$. Then $\bar{u} \in \ell^\infty(\mathbb{Z})$.

Let us consider $\lambda \in \sigma(\tilde{H}_x^{p,i})$ and u like in the Floquet theorem. Then $\tilde{H}_x^{p,i}u = \lambda u \Longrightarrow \tilde{H}_x^{q,i}\bar{u} = \lambda\bar{u}$, and for all n, $u(n+\tilde{N}_p) = \rho u(n) \Longrightarrow \bar{u}(n+\tilde{N}_q) = (1/\rho)\bar{u}(n)$ with $|1/\rho| = |\rho| = 1$ ($\tilde{N}_p = \tilde{N}_q$). We thus get $\sigma(\tilde{H}_x^{p,i}) \subset \sigma(\tilde{H}_x^{q,i})$. In the same manner, we establish the equality of the spectra and the proof is complete. □

We formulate our main results in a theorem.

Theorem 5.8. *Let p be a non-negative integer, and H_x^p a p-sparse N-periodic Schrödinger operator. Then*

(i) $\sigma(H_x^p) = \Sigma^p$ *for all* $x \in X$;
(ii) *If* $p \equiv 0 \bmod (N)$, *then* $\Sigma^p = \cup_{i=0}^{N-1}[f(i)-2, f(i)+2]$;
(iii) *If* $p \not\equiv 0 \bmod (N)$, *then* $\Sigma^p = \Sigma^{p \bmod (N)} = \Sigma^{N-(p \bmod (N))}$;
(iv) Σ^p *is purely absolutely continuous and it is composed of N bands.*

The nature of the spectrum of a family of p-sparse N-periodic Schrödinger operators does not change with p. It is always purely absolutely continuous and always composed of N bands. Moreover, the spectrum itself can change according to the values of p. Notice that Σ^p is exactly the spectrum of the classical N-periodic Schrödinger operator when $p \equiv 1 \bmod (N)$.

5.3. *Examples*

Example 5.3.1. *The case of p-sparse 1-periodic Schrödinger operators*

Let us consider $X = \mathbb{Z}/1\mathbb{Z}$ (i.e. $X = \{\bar{0}\}$), and $V_0(n) = \alpha$ for all n. Then Σ^p is absolutely continuous and composed of a unique band for any non-negative integer p.

$$\Sigma^p = [\alpha - 2, \alpha + 2] \ .$$

Example 5.3.2. *The case of p-sparse 2-periodic Schrödinger operators*

Let us consider $X = \mathbb{Z}/2\mathbb{Z}$. The potential takes two values α and β. We suppose $\alpha < \beta$. Then

$$\Sigma^p = \begin{cases} \Sigma^1 & \text{if } p \text{ is even} \\ \Sigma^2 & \text{if } p \text{ is odd.} \end{cases}$$

We know exactly the form of these spectra.

$$\Sigma^1 = \left[\frac{\alpha+\beta-\sqrt{16+(\alpha-\beta)^2}}{2}, \alpha\right] \bigcup \left[\beta, \frac{\alpha+\beta+\sqrt{16+(\alpha-\beta)^2}}{2}\right]$$

$$\Sigma^2 = [\alpha-2, \alpha+2] \bigcup [\beta-2, \beta+2] \ .$$

Thus, there exist exactly two disjoint bands in the spectrum when p is odd, but in the case where p is even, one or two bands can appear.

In particular when $\alpha = -\beta$, with $\beta > 0$, we always have exactly two bands in the spectrum.

$$\Sigma^1 = \left[-\sqrt{4+\beta^2}, -\beta\right] \bigcup \left[\beta, \sqrt{4+\beta^2}\right]$$

$$\Sigma^2 = [-\beta-2, -\beta+2] \bigcup [\beta-2, \beta+2] \ .$$

In this case, notice that the spectrum is symmetric with respect to the origin.

Example 5.3.3. *The case of p-sparse 3-periodic Schrödinger operators*

Let us consider $X = \mathbb{Z}/3\mathbb{Z}$. Then

$$\Sigma^p = \begin{cases} \Sigma^1 & \text{if } p \equiv 1 \text{ or } 2 \bmod (3) \\ \Sigma^3 & \text{if } p \equiv 0 \bmod (3). \end{cases}$$

We know the form of these spectra:

$$\Sigma^1 = \overline{\{\lambda \in \mathbb{R} \ / \ [(\lambda - f(0))(\lambda - f(1))(\lambda - f(2)) - (\lambda - f(0)) - (\lambda - f(1))} \\ \overline{- (\lambda - f(2))]^2 \leq 4\}}$$

$$\Sigma^3 = [f(0)-2, f(0)+2] \bigcup [f(1)-2, f(1)+2] \bigcup [f(2)-2, f(2)+2]$$

and there exist at most three bands. Notice that the three values of f play the same role in Σ^p.

If we suppose $f(0) = 0$, $f(1) = \alpha$ and $f(2) = -\alpha$ with $\alpha > 0$, then we immediately have

$$\Sigma^3 = [-\alpha-2, -\alpha+2] \bigcup [-2, 2] \bigcup [\alpha-2, \alpha+2] \ .$$

Moreover, $\Sigma^1 = \overline{\{\lambda \in \mathbb{R} \ / \ [\lambda^3 - (3+\alpha^2)\lambda + 2][\lambda^3 - (3+\alpha^2)\lambda - 2] \leq 0\}}$, and we can show there exist three positive reals $\lambda_1 < \lambda_2 < \lambda_3$, such that

$$\Sigma^1 = [-\lambda_3, -\lambda_2] \bigcup [-\lambda_1, \lambda_1] \bigcup [\lambda_2, \lambda_3] \ .$$

Thus Σ^1 is always composed of 3 disjoint bands whereas Σ^3 can have 1 or 3 bands according to α is strictly greater than 4 or not. Anyway, the spectrum is again symmetric with respect to the origin.

6. The Random Case

Let us consider $X = S^{\mathbb{Z}}$, where S is a Borel subset of $\mathbb{R}$, $\mathcal{B}$ the σ-algebra generated by the cylinder sets, i.e. by sets of the form $\{x \,/\, x_{n_1} \in A_1, \ldots, x_{n_q} \in A_q\}$ for $n_1, \ldots, n_q \in \mathbb{Z}$ and $A_1, \ldots, A_q$ Borel sets in $\mathbb{R}$. We consider a sequence $(V(n))_{n\in\mathbb{Z}}$ of random variables. The sequence of potentials is given by $V_x(n) = V(n)(x)$ for all x in X (we say V_x is a realization).

In all this part, we will suppose the $V(n)$ are independent identically distributed random variables of product distribution μ and of same law $r(.)$. Remark that the dynamical system $\mathcal{T} = (T, X, \mathcal{B}, \mu)$, where T is the shift operator on X, is ergodic. The discrete one-dimensional Schrödinger operator H_x associated with the sequence of potentials $(V_x(n))_{n\in\mathbb{Z}}$, is referred to as the *Anderson model* (see [5, Chap. 9], or [9]). According to this, the p-sparse Schrödinger operator H_x^p is called a *p-sparse Anderson model.*

Proposition 6.1. *Let $(H_x^p)_{x\in X}$ be a p-sparse Anderson model. Then $(\tilde{H}_x^{p,i})_{x\in X}$, for all $i \in \{0, \ldots, p-1\}$, is an Anderson model.*

Proof. Each $\tilde{V}_x^{p,i}$ is a realization of a sequence of the random variables $(V(np+i))_{n\in\mathbb{Z}}$. This sequence is also an independent identically distributed random variables sequence, whose common product distribution is again μ and law is $r(.)$. □

This proposition directly leads to a more precise result as Theorem 3.7.

Corollary 6.2. *Under the assumptions of Proposition* 6.1, *the spectrum Σ^p of μ-almost all operators H_x^p is the union of p compact subsets of $\mathbb{R}$. More precisely*

$$\Sigma^p = \bigcup_{i=0}^{p-1} \Sigma^{p,i} \quad \text{where} \quad \Sigma^{p,i} = \sigma(\tilde{H}_x^{p,i}) \quad \mu - p.p. \tag{6.1}$$

Moreover, if $\varepsilon \in \{pp, ac, sc\}$, and if Σ_ε^p (respectively $\Sigma_\varepsilon^{p,i}$) is the ε-component of μ-almost all operators H_x^p (resp. $\tilde{H}_x^{p,i}$), then

$$\Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \Sigma_\varepsilon^{p,i} \,. \tag{6.2}$$

Proof. It is a consequence of a theorem given by Kunz and Souillard in [14] and extended by Kirsch and Martinelli in [12]. By Proposition 6.1, i being fixed in $\{0, \ldots, p-1\}$, $(\tilde{H}_x^{p,i})_{x\in X}$ is an Anderson model. Thus μ-almost all operators $\tilde{H}_x^{p,i}$ have the same spectrum and spectral components. Theorem 3.7 concludes the proof. □

Moreover, an analogue of the Kotani–Simon theorem can be stated.

Theorem 6.3. *Let $(H_x^p)_{x\in X}$ be a p-sparse Anderson model. Then $\sigma_{ac}(H_x^p) = \emptyset$ for μ-almost all x.*

Proof. By Proposition 6.1, $(\tilde{H}_x^{p,i})_{x\in X}$ is an Anderson model. The theorem of Kotani–Simon establishes that for μ-almost all x, $\sigma_{ac}(\tilde{H}_x^{p,i}) = \emptyset$ (for more details, we refer the reader to [13] and [17]). Corollary 6.2 concludes the proof. □

According to some peculiar properties of the common density function $r(.)$, several results can be deduced. Firstly we can state an analogue of the Kunz and Souillard theorem (see for instance [14, 9] or [5, 1, 6]).

Theorem 6.4. *Let $(H_x^p)_{x\in X}$ be a p-sparse Anderson model. We suppose that the common density function of $(V(n))_n$ is a non-negative function such that there exists a real $0 < \lambda < 1$ with*

$$\sup_{t\in\mathbb{R}} |r(t)(1+|t|)^{1+\lambda}| < +\infty \ . \tag{6.3}$$

Then, with probability 1, the spectrum is pure point and equal to

$$\Sigma^p = [-2,2] + \mathrm{Supp}(r) \ . \tag{6.4}$$

Moreover all eigenvectors are exponentially localized.

Remark 6.1. Supp(r) denotes the support of the function r. If A and B are two subsets of $\mathbb{R}$, then $A + B = \{a + b \ ; \ a \in A \ \text{ and } b \in B\}$.

Remark 6.2. According to Property (6.3), r is a bounded function.

Proof. By Proposition 6.1, we can apply the Kunz and Souillard theorem, given in [14], to each family $(\tilde{H}_x^{p,i})_{x\in X}$. For μ-almost all x, the spectrum of $\tilde{H}_x^{p,i}$ is pure point and equal to $[-2,2] + \mathrm{Supp}(r)$. Applying Corollary 6.2 completes the proof. □

In the particular case where the common density function $r(.)$ is continuous with compact support, we get Corollary 6.5.

Corollary 6.5. *Let $(H_x^p)_{x\in X}$ be a p-sparse Anderson model. We suppose that the common density function of $(V(n))_n$ verifies the following conditions:*

(a) *r is a continuous function on $\mathbb{R}$;*
(b) *$r(t) = 0 \Longleftrightarrow t \notin]\alpha,\beta[\subset \mathbb{R}$.*

Then the spectrum is μ-almost surely pure point and equal to

$$\Sigma^p = [\alpha - 2, \beta + 2] \ . \tag{6.5}$$

Moreover, all eigenvectors are exponentially localized.

Proof. Support of $r(.)$ is $[\alpha,\beta]$ and r verifies Relation (6.3) for any λ in $]0,1[$. Proof is complete in applying Theorem 6.4. □

On the other hand, if the random variables $V(n)$ are Bernoulli distributed, an analogue of the Carmona, Klein and Martinelli theorem is verified (see [9] for instance).

Theorem 6.6. *Let $(H_x^p)_{x\in X}$ be a p-sparse Anderson model. We suppose that the sequence $(V(n))_{n\in\mathbb{Z}}$ admits a Bernoulli distribution, that is to say:*

$$V(n)(x) = \begin{cases} 0 & \text{with probability } \rho \\ \alpha & \text{with probability } 1-\rho, \text{ and } \alpha \text{ real.} \end{cases}$$

Then the spectrum is μ-almost surely pure point and equal to

$$\Sigma^p = [-2,2] \bigcup [\alpha - 2, \alpha + 2] \ . \tag{6.6}$$

Moreover, all eigenvectors are exponentially localized.

Proof. Using the same arguments as in the proof of Theorem 6.4 and according to the Carmona, Klein and Martinelli theorem given in [9], we obtain the proof. □

Remark 6.3. Let us mention that in the special cases corresponding to Theorems 6.4 and 6.6, and to Corollary 6.5, neither the spectrum nor its nature changes with p.

7. Sparse Schrödinger Operators with Substitutional Potentials

The class of almost periodic potentials lies between periodic and random cases. We study here the subclass of substitutional potentials. In the first part, we recall some elementary results in substitutional sequences. For more details, we refer the reader to [16].

7.1. *Substitutional dynamical systems*

Let us consider a finite set $\mathcal{A} = \{0, \ldots, r-1\}$ called an *alphabet*. We denote by $\mathcal{A}^{\mathbb{Z}}$ the set of all bi-infinite sequences of letters from $\mathcal{A}$. A *word* is a finite sequence of letters. We consider a *substitution* ξ which associates to any letter a in $\mathcal{A}$, a word $\xi(a)$. Moreover ξ will be supposed *primitive*, which means that there exist a non-negative integer k such that for all pairs of letters a and b in $\mathcal{A}$, the word $\xi^k(a)$ contains the letter b.

Under the primitivity condition, ξ admits fixed-points, that is to say there exist bilateral sequences $w = \ldots\ w_{-1}\ w_0\ w_1\ \ldots$ in $\mathcal{A}^{\mathbb{Z}}$ such that $\xi(w) = w$ (see [7, 16]). Such a fixed-point w is called a *substitutional sequence*. It is an *almost periodic* sequence, which means that every word of w occurs in w with bounded gaps (the bound depending on the word). We denote by T the shift operator on $\mathcal{A}^{\mathbb{Z}}$. A topological dynamical system can be assigned in a natural way to the substitutional sequence w. Precisely

$$X(\xi) = \overline{\{T^k w \ ; \ k \in \mathbb{Z}\}} \ , \tag{7.1}$$

where the closure is in the strong sense in $\mathcal{A}^{\mathbb{Z}}$. $X(\xi)$ is a compact metrizable set. The restriction of T to $X(\xi)$ is again denoted by T. The pair $(X(\xi), T)$ is

a topological dynamical system. Moreover it is minimal. Notice that under the primitivity condition on ξ, any fixed-point w of ξ generates the same dynamical system. From now on, a fixed-point w is given.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $X(\xi)$. Under primitivity of ξ, there exists a unique T-invariant probability measure μ on $X(\xi)$, which turns out to be ergodic (for more details see [16]). Thus $\mathcal{T} = (T, X(\xi), \mathcal{B}, \mu)$ is a strictly ergodic dynamical system. $\mathcal{T}$ is called the *dynamical system generated by substitution* ξ.

By strict ergodicity of $\mathcal{T}$, we can apply Theorem 4.1 for a given non-negative integer p, we find a partition of $X(\xi)$ into $\delta(p)$ parts. We always choose X_0 to be the member of the partition containing w. When $\delta(p) = p$, we deduce immediately from Corollary 4.3, the strict ergodicity of $\mathcal{T}_0^p = (T^p_{|X_0}, X_0, \mathcal{B}_{|X_0}, \mu_0)$.

Proposition 7.1. *If p is a non-negative integer such that $\delta(p) = 1$, then $\mathcal{T}^p = (T^p, X(\xi), \mathcal{B}, \mu)$ is a minimal ergodic dynamical system.*

Proof. According to [7], T^p is minimal if and only if it is ergodic. □

Now, if $B = b_0 \dots b_{j-1}$ is a word of letters from $\mathcal{A}$, then j is called the *length* of B and is denoted by $|B|$. When for any letter a of $\mathcal{A}$, the length of $\xi(a)$ is equal to ℓ, where ℓ is a non-negative integer, the substitution ξ is said to have *constant length* or *uniform length*. Otherwise, it has *non constant length*. In the case where ξ is a substitution with constant length, we can say more about the dynamical system $\mathcal{T}^p$ (we refer the reader to [7]).

Proposition 7.2. *Let us consider a substitution ξ with a constant length ℓ, and a non-negative integer p.*

(i) *If $\delta(p) = 1$, then there exists a finite alphabet $\hat{\mathcal{A}}$ and a primitive substitution η with constant length equal to ℓ on $\hat{\mathcal{A}}$, such that the dynamical system generated by the substitution η is isomorphic to $\mathcal{T}^p$.*

(ii) *If $\delta(p) = p$, then there exists a finite alphabet $\hat{\mathcal{A}}$ and a primitive substitution η with constant length equal to ℓ on $\hat{\mathcal{A}}$, such that the dynamical system generated by the substitution η is isomorphic to $\mathcal{T}_0^p$.*

(iii) *In particular when $p = \ell^m$, with $m \geq 1$, we know $\delta(\ell^m) = \ell^m$. Moreover $\hat{\mathcal{A}} = \mathcal{A}$, $\eta = \xi$, thus $\mathcal{T}_0^p$ is isomorphic to $\mathcal{T}$.*

Remark 7.1. The alphabet $\hat{\mathcal{A}}$ and the substitution η, just as the isomorphism between $\mathcal{T}^p$ (respectively $\mathcal{T}_0^p$) and the dynamical system generated by η, are explicitly constructed in the proof of M. F. Dekking (see [7]).

7.2. *Properties of the sparse Schrödinger operators*

We only suppose for instance that ξ is a primitive substitution. Let us consider the potential $(V_x(n))_{n\in\mathbb{Z}}$ given by

$$V_x(n) = f(T^n x) \ , \ \forall\, n \in \mathbb{Z} \tag{7.2}$$

where f is a real-valued bounded measurable application on $X(\xi)$. Then H_x^p is called a *p-sparse Schrödinger operator with substitutional potential.* In the theory

of Schrödinger operators we usually consider f verifying

$$f(x) = v(x_0) \ , \tag{7.3}$$

where v is a finite real-valued map from $\mathcal{A}$ and x_0 is the first component of x. Moreover v is chosen so that the resulting sequence of potential values $V_w = (v(w_n))_{n\in\mathbb{Z}}$ is aperiodic.

According to Proposition 4.6, we know that if $\delta(p) = p$, then Σ^p and its components are the unions of p compact sets, which are respectively the spectra and the spectral components of the associated operators on the dynamical system $\mathcal{T}_0^p$. In other words,

$$\Sigma^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}^{p,i} \quad \text{and} \quad \Sigma_\varepsilon^p = \bigcup_{i=0}^{p-1} \tilde{\Sigma}_\varepsilon^{p,i} \ ,$$

where $\varepsilon \in \{pp, ac, sc\}$, $\tilde{\Sigma}^{p,i} = \sigma(\tilde{H}_x^{p,i})$ for any $x \in X_0$, and $\tilde{\Sigma}_\varepsilon^{p,i} = \sigma_\varepsilon(\tilde{H}_x^{p,i})$ for μ_0-almost every x in X_0. By Proposition 7.1, we find a similar result when $\delta(p)$ equals 1.

Proposition 7.3. *If p is a non-negative integer such that $\delta(p) = 1$, then*

(i) *there exists a non-empty compact set of $\mathbb{R}$, denoted by $\tilde{\Sigma}^{p,0}$ which is the spectrum of $\tilde{H}_x^{p,0}$ for any $x \in X(\xi)$, and such that*

$$\Sigma^p = \tilde{\Sigma}^{p,0} \ ; \tag{7.4}$$

(ii) *for $\varepsilon \in \{pp, ac, sc\}$ and if $\tilde{\Sigma}_\varepsilon^{p,0}$ denotes the ε-component of the spectrum of $\tilde{H}_x^{p,0}$ for μ-almost all x in $X(\xi)$, we see that*

$$\Sigma_\varepsilon^p = \tilde{\Sigma}_\varepsilon^{p,0} \ . \tag{7.5}$$

Proof. According to (4.3), (4.4) and Proposition 7.1, and since $X_0 = X(\xi)$, the proof is complete. □

This proposition means that in the case where $\delta(p)$ is equal to 1, Σ^p is exactly the spectrum of a certain family of Schrödinger operators $(\tilde{H}_x^{p,0})_{x\in X(\xi)}$ whose potentials are given for any x in $X(\xi)$ by

$$\tilde{V}_x^{p,0}(n) = V_x(np) = v(x_{np}) \ , \ \forall n \in \mathbb{Z} \ . \tag{7.6}$$

In the case where $\delta(p)$ equals p, we have to study the p families of the associated operators $(\tilde{H}_x^{p,i})_{x\in X_0}$, with $0 \le i < p$, whose potentials are given for any x in X_0 and for $0 \le i < p$, by

$$\tilde{V}_x^{p,i}(n) = V_x(np+i) = v(x_{np+i}) \ , \ \forall n \in \mathbb{Z} \ . \tag{7.7}$$

We are now able to state, in the extremal cases $\delta(p) = 1$ and $\delta(p) = p$, a theorem concerning the absolutely continuous component of Σ^p.

Theorem 7.4. *Let ξ be a primitive substitution on the alphabet $\mathcal{A}$ and p a non-negative integer. If p is such that $\delta(p) = 1$ (respectively, $\delta(p) = p$), then there is no absolutely continuous spectrum μ-almost surely (respectively, μ_0-almost surely).*

Proof. We first consider the case $\delta(p) = 1$. By (7.6) and Proposition 7.3, we have to study a new family of Schrödinger operators $(\tilde{H}_x^{p,0})_{x\in X(\xi)}$, associated with the strictly μ-ergodic dynamical system $\mathcal{T}^p$. But aperiodicity of V_w means that the topological support of μ is not a finite set. Thus $\tilde{V}_w^{p,0}$ is not periodic, and we can apply the Kotani theorem to the family $(\tilde{H}_x^{p,0})_{x\in X(\xi)}$ (see for instance [13]): for μ-almost every x in $X(\xi)$, $\tilde{H}_x^{p,0}$ does not admit any absolutely continuous part in its spectrum.

For the second case $\delta(p) = p$, we use (7.7) and the fact that the topological support of μ_0 is finite if and only if it is the same for the topological support of μ, in view to apply the Kotani theorem to the p families $(\tilde{H}_x^{p,i})_{x\in X_0}$ with $0 \leq i < p$. Proposition 4.6 completes the proof. □

When the substitution ξ is primitive and has constant length, Proposition 7.2 can be applied, and means, under the condition $\delta(p) = 1$ or p, that each operator $\tilde{H}_x^{p,i}$ is a Schrödinger operator with substitutional potential. Moreover, when $p = \ell^m$, the corresponding substitution is again ξ, and the nature of the spectrum of these ℓ^m-sparse Schrödinger operators is the same as in the classical case of one-dimensional discrete Schrödinger operators with the same substitutional potential. We can say more: the nature of the spectrum Σ^{ℓ^m} does not change with $m \geq 1$.

We will illustrate these results by examples.

7.3. *Examples*

Example 7.3.1. *The period-doubling substitution*

We consider the alphabet $\mathcal{A} = \{0, 1\}$. The period doubling substitution is defined by

$$\begin{aligned} \xi: \quad 0 &\longrightarrow 01 \\ 1 &\longrightarrow 00 \,. \end{aligned}$$

It is a primitive substitution with constant length equal to 2. We can choose the fixed-point $w = \lim_{\leftarrow} \xi^{2n}(0) \lim_{\rightarrow} \xi^{2n}(0)$, i.e.

$$w = \ldots\ 0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,\overline{0}\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\ \ldots \quad .$$

By primitivity of ξ, $X(\xi)$ is generated by any fixed-point, and the dynamical system generated by ξ is strictly ergodic.

Moreover, the decomposition function can be calculated (for more details see [7]):

$$\begin{aligned} \delta(2^n) &= 2^n \quad \forall n \geq 1 \\ \delta(m) &= 1\,, \quad \text{if } m \text{ is odd} \\ \delta(2^n m) &= 2^n\,, \quad \forall n \geq 1\,, \quad \forall\, m \text{ odd}\,. \end{aligned}$$

According to Theorem 7.4, the following proposition is deduced.

Proposition 7.5. *Let us consider the period doubling substitution on* $\mathcal{A} = \{0, 1\}$. *Then*

(i) *for any (non-negative) odd integer* p, Σ^p *is the spectrum of a Schrödinger operator with substitutional potential* $\tilde{H}_x^{p,0}$, *for any* $x \in X(\xi)$, *and* $\Sigma_{ac}^p = \emptyset$;

(ii) *for any non-negative integer* n, Σ^{2^n} *is a Cantor set of zero Lebesgue measure, and for* μ*-almost all* x *in* X, *it is purely singular continuous.*

Proof. (i) is a direct consequence of Theorem 7.4. For (ii), we remark that each $\tilde{H}_x^{p,i}$ is a Schrödinger operator with substitutional potential generated by the period doubling substitution. But we know, according to [2] (see also [4, 15, 18]), that the spectrum of every associated operator on X_0 is a Cantor set of zero Lebesgue measure, and it is for μ_0-almost x in X_0 purely singular continuous. So is Σ^{2^n}. □

Example 7.3.2. *The Thue-Morse substitution*

We consider the alphabet $\mathcal{A} = \{0, 1\}$. The Thue–Morse substitution is defined by

$$\begin{aligned} \xi: \quad 0 &\longrightarrow 01 \\ 1 &\longrightarrow 10 \,. \end{aligned}$$

It is a primitive substitution with constant length equal to 2. We can choose the fixed-point $w = \underline{\lim}\, \xi^{2n}(0) \underrightarrow{\lim}\, \xi^{2n}(0)$, i.e.

$$w = \ldots\ 0\ 1\ 1\ 0\ \overline{0}\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ \ldots \quad .$$

By primitivity of ξ, $x(\xi)$ is generated by any fixed-point, and the dynamical system generated by ξ is strictly ergodic. Moreover, the decomposition function can be calculated:

$$\begin{aligned} \delta(2^n) &= 2^n \,, \quad \forall n \geq 1 \\ \delta(m) &= 1 \,, \quad \text{if } m \text{ is odd} \\ \delta(2^n m) &= 2^n \,, \quad \forall n \geq 1 \,, \quad \forall\ m \text{ odd} \,. \end{aligned}$$

According to Theorem 7.4, we find an analogue of Proposition 7.5:

Proposition 7.6. *Let us consider the Thue–Morse substitution on* $\mathcal{A} = \{0, 1\}$. *Then*

(i) *for any (non-negative) odd integer* p, Σ^p *is the spectrum of a Schrödinger operator with substitutional potential* $\tilde{H}_x^{p,0}$, *for any* $x \in X(\xi)$, *and* $\Sigma_{ac}^p = \emptyset$;

(ii) *for any non-negative integer* m, Σ^{2^m} *is a Cantor set of zero Lebesgue measure, and for* μ*-almost all* x *in* X, *it is purely singular continuous.*

8. Conclusion

We can now compare the three special cases of p-sparse Schrödinger operators treated in this paper. When the potential is a sequence of independent identically distributed random variables, neither the nature of the spectrum nor its location change with p: the spectrum of H^p is exactly the one of the corresponding Schrödinger operator $H = H^1$.

On the other side, if the potential is N-periodic, then the nature of the spectrum does not change with p, just as the number of its bands (there are always N bands). But Σ^p (the spectrum of H^p), is the same as the spectrum of the classical N-periodic Schrödinger operator only in the case of $p \equiv 1 \bmod (N)$ or $p \equiv N-1 \bmod (N)$.

The case of the substitutional potentials, which lies between the two others, is more complicated because linked with the decomposition function $\delta(p)$. In fact we only know that there exists no absolutely continuous part in the spectrum of H^p if $\delta(p) = 1$ or p. Moreover, if the primitive substitution has a constant length ℓ and if $p = \ell^n$, then the nature of Σ^p is the one of the spectrum of the corresponding substitutional Schrödinger operator H.

Finally, in these cases (random, periodic and substitutional), the spectral behavior of p-sparse Schrödinger operators is similar to the one of the corresponding Schrödinger operators for all p when the potential is random or periodic, and for $\delta(p) = 1$ or p when it is substitutional. We could conjecture similar results for sparse Schrödinger operators with limit periodic or quasi-periodic potentials. Such a study will surely leads us to a more accurate understanding of these random operators.

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