On exteriority notions in book embeddings and treewidth

Nicolas Baudru, Severine Fratani

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Nicolas Baudru[1] and Séverine Fratani[1]

Aix Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France
{nicolas.baudru,severine.fratani}@lis-lab.fr

Abstract. Book embeddings generalize planar embeddings to a space formed by several half-planes sharing their boundary. While several measures on planar graphs allow to define treewidth-bounded classes of planar graphs, no such results exist for book embeddings. Indeed, many of these measures rely on the notion of exteriority that cannot be simply generalized to books due to their complex topology. In this paper, we first propose a notion of exteriority for book embeddings, and then, define an outerplanar-like measure: a book embedding is k-outeredge if the distance from its edges to the outside is at most k. We exhibit a large class of k-outeredge book embeddings that is treewidth-bounded by $\Omega(2^k)$ and $\mathcal{O}(p^k)$, for a fixed number p of half-planes. The lower bound comes from a nice connection with formal verification results.

1 Introduction

1.1 Context

Book embeddings [1,2] (also called stack layouts) generalize planar graphs to more than one plane. A p-book is a topological space that consists of p half-planes, called *pages*, glued together along their boundary, called *spine*. A graph is *embedded* in a book if every vertex is drawn on the spine and every edge is wholly drawn on a single page without crossing another edge. The study of book embeddings has been motivated by several areas of computer science as VLSI theory, multilayer printed circuit boards and sortings with parallel stacks. In these contexts, a well-studied problem consists in embedding a graph into a book using a minimum number of pages [1,3,4,5].

Another typical problem in graph theory is to determine an upper bound on the treewidth of a class of graphs [6,7]. Loosely speaking, the treewidth measures how far a graph is from a tree. It is often used as a parameter in the parametrized complexity analysis of graph algorithms. Also, it plays an impo,rtant part in formal verification for the model checking of monadic-second-order (MSO) formulas on graphs through Courcelle's classic theorem [8]. Several classes of planar graphs are known to have bounded treewidth when they have a bounded radius, width (or gauge), depth or outerplanarity (see [9,6]). All these measures describe in a different way the maximum distance from vertices (or faces) of a planar graph to the outer face. Such an important result states that k-outerplanar graphs are treewidth-bounded by $3k-1$ [6].

In this paper, we aim to give a similar result for classes of book embeddings.

1.2 Motivation and application

Identifying classes of book embedding with bounded treewidth (or other related parameters such as the bandwidth or the size of a separator of a graph [10,11]) is of great interest in computational theory. We give here some examples.

Classes of book embeddings, called p-page graphs and p-pushdown graphs [12], naturally arise as computation graphs of Turing machines. This observation was used as early as 1986 for proving that 3-pushdown graphs (or k-page graphs) have sublinear separators if, and only if, a one-tape nondeterministic Turing machine can simulate a two-tape machine in subquadratic time [13]. This result constitutes the first example where a graph problem is shown to be equivalent to a problem in computational complexity. Other results in the same spirit can be found in [14,15].

Another interesting line of research concerns the identification of restricted classes of multi-pushdown automata with a decidable emptiness problem [16,17,18] and [19,20,21,22]. Proving that the computation graphs of these machines are treewidth-bounded classes of multi-nested words [23] (multi-nested words are close to multi-pushdown graphs of degree 3 in [14]), the authors of [24] give a general proof for most of these decidability results through Courcelle's classic theorem.

A substantial and tedious part of the proof of [24] consists in determining an upper bound on the treewidth of each subclass of multi-nested words involved by each considered restriction on multi-pushdown machines. Interestingly, several of these subclasses are *multi-outeredge covered-spine book embeddings* that we define in this paper, and for which we bound the treewidth. This suggests that our result can be helpful as it turns a treewidth calculation problem into an inclusion problem that we believe easier. As a concrete example, we prove that k-phases nested words are particular k-outeredge covered-spine book embeddings.

1.3 Overview and main results

In Sec. 2 we recall basic notions on graphs and book embeddings. The three next sections present the two main contributions of this paper:

A notion of exteriority for the book topology Outerplanarity-like measures generally rely on a notion of exteriority in the considered topology. For books, this notion cannot be easily extended from that of the planar case. For instance, contrary to the planar case, we cannot define the exterior (or the outer face) as the unbounded connected component of the complement of the book embedding, since there is often only one connected component.

In Sec. 3, we introduce a face-like notion for book embeddings called *regions*. A cycle ρ of an embedding splits each half-plane of the book into connected components that we call *facets*. The region $\mathcal{R}$ enclosed by ρ is a suitable collection of facets that defines a bounded and connected space "delimited" by ρ. A point of the book is *internal* if it belongs to a region of the book embedding. A vertex or an edge of the embedding is internal if it consists of internal points. At the end of Sec. 3, we define the *k-outeredge measure*, which is a variant of the k-outerplanarity based on the external edges of a book embedding. This constitutes the first important contribution of this paper.

A treewidth-bounded subclass of k-outeredge book embeddings Contrary to the class of k-outerplanar graphs, the class of k-outeredge book embeddings is not treewidth-bounded. In Sec. 4, we define the *covered-spine* property, which requires that all internal spine points of a book embedding G are vertices or belong to edges of G. We prove that any graph G with degree d that admits a k-outeredge covered-spine p-book embedding has a treewidth at most $\frac{3}{2}d(p-1)^k$ (Theorem 17). The computation of this upper bound follows a technical and non trivial adaptation of the method used in [6] for planar graphs. This result is the second main contribution of this paper. We improve it in Sec. 5 by proving that any graph that admits a k-outeredge covered-spine p-book embedding is the minor of a graph with degree 3 that admits a $(k+1)$-outeredge covered-spine p-book embedding . Thereby, we get a new bound that no longer depends on the degree (Theorem 35).

As mentioned above, the class of multi-outeredge covered-spine book-embeddings is large enough to include several known restrictions of multi-nested words. In particular, we show in Sec. 6 that k-phase nested words are particular k-outeredge covered-spine book-embeddings. From this connection, a lower bound on the treewidth in $\Omega(2^k)$ (Theorem 45) is established. It is noteworthy that our lower bound comes from a decidability result on multi-pushdown automata [17]. This way, this paper sets another bridge between book embeddings and computational theory. Finally, in the discussion of Sec. 7, we show why our outer measure relies on external edges rather than on external vertices.

2 Preliminary definitions

2.1 Graph, minor, planarity and treewidth

As usual, a graph G is a structure (V, E) where V is a finite set of *vertices* and $E \subseteq V \times V$ is a set of *edges*. Throughout this paper, we consider undirected and simple graphs only, i.e. E is symmetric and irreflexive. The *degree* of a vertex v of G is the number of edges incident to v. The degree of G, denoted by $\deg(G)$, is the maximum degree of its vertices. A (simple) *path* in G is a finite sequence $\alpha = (v_0, v_1, \ldots, v_n)$ of pairwise distinct vertices of G such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i \leq r-1$. If $v_0 = v_r$, then α is called a (simple) *cycle*. A *forest* is a simple cycle-free graph and a *tree* is a connected forest. A *subgraph* of G is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. A forest $T = (V, E')$ is a *maximal spanning forest* of G if T is a subgraph of G such that, for all $e \in E - E'$, the graph $(V, E' \cup \{e\})$ is no longer a forest. A graph G is a *minor* of a graph G' if G is obtained from G' by a series of vertex deletions, edge deletions or edge contractions, where an edge contraction means merging two adjacent vertices v and w (so, all vertices adjacent to v and w become adjacent to the merged vertex). Clearly, any subgraph of G is also a minor of G.

A tree decomposition of a graph G is a tree T with a family $(V_w)_{w \in W}$ of subsets of vertices of G satisfying the following properties: (i) Every node of G belongs to at least one V_w; (ii) For every edge $e = (v, v')$ of G, there is a subset V_w containing both v and v'; (iii) If V_w and $V_{w'}$ contain a vertex v of G, then for all nodes w'' on the (unique) path from w to w' in T, $V_{w''}$ contains v as well. The *width* of a tree decomposition is the size of its largest set V_w minus one. The *treewidth* $\mathrm{tw}(G)$ is the minimum width among all possible tree decompositions of G. The treewidth of a class $\mathcal{C}$ of graphs is $\mathrm{tw}(\mathcal{C}) = \max\{\mathrm{tw}(G) \mid G \in \mathcal{C}\}$. Note that the treewidth of a tree is 1.

The following well-known result gives a connection between the notions of minor and treewidth.

Proposition 1 (see e.g. [6]). *If a graph G is a minor of a graph G', then* $\mathrm{tw}(G) \leq \mathrm{tw}(G')$.

A *planar embedding* of a graph G is a drawing of G in the plane such that its edges intersect at their endpoints only. A *face* is a (topological) connected component of the complement of (the drawing of) G. The *outer face* is the unique unbounded face of the embedding. A vertex or an edge is *external* if it is on the boundary of the outer face. Otherwise it is internal. A planar embedding is *(1-)outerplanar* if all its vertices are external; it is *k-outerplanar* if deleting all the external vertices gives a $(k-1)$-outerplanar embedding. Outerplanar embeddings have been widely studied in the literature. In particular, it is proved in [6, Theorem 83] that every graph that admits a k-outerplanar embedding has treewidth at most $3k-1$. The reader can refer to [25] for a complete introduction to graph theory.

2.2 Book embedding

Book embeddings were introduced by Kainen and Ollmann [1,2]. We slightly generalize their definition, allowing edges to be drawn on the spine. This generalization preserves the major existing results, in particular those dealing with the minimal number of pages (called *book thickness*) required to embed a graph in a book [3,4,5].

Definition 2. *An* embedding *of a graph G in a book $\mathcal{B}$ is a representation (drawing) of G in $\mathcal{B}$, in which vertices are associated to points of $\mathcal{B}$ and edges are associated to simple arcs in such a way that:*

1. *all its vertices are drawn on the spine at distinct points;*
2. *the endpoints of the arc associated to an edge (v, v') are the points associated to v and v';*

3. *an arc is either wholly drawn on the spine (such an edge is called a* spine edge*), or else wholly drawn on a single page (such an edge is called a* page edge*). In the latter case only the endpoints belong to the spine;*
4. *an arc includes no point associated with vertices, except at its endpoints;*
5. *two arcs in the same page can intersect only at their endpoints.*

A subembedding *of an embedding of a graph G is the embedding of a subgraph G' of G obtained by erasing from the embedding of G all the vertices and all the arcs that are not in the subgraph G'.*

Throughout this paper, we consider embeddings with at least two pages. We suppose that the spine, denoted by ℓ, is endowed with a natural linear order $<$ on the points of ℓ. When an embedding of a graph $G = (V, E)$ is fixed, we use an abuse of language and notation and we identify G with its book embedding. In this way, $v \in V$ (resp. $e \in E$) refers to both a vertex (resp. an edge) of G and its associated point (resp. arc) in the embedding. We denote by x_e and y_e the endpoints of an edge e in the embedding, and always suppose that $x_e < y_e$. Let e° denote the arc $e - \{x_e, y_e\}$ in $\mathcal{B}$. Similarly, a simple path π of G is identified with the corresponding curve in $\mathcal{B}$. We often write $e \subset \pi$ to mean that the arc e is a part of the curve π.

Let $G = (V^G, E^G)$ be a graph embedded in a p-book. The set of edges E can be partitioned into the set E_S^G of spine edges and the set E_P^G of page edges. Clearly, page edges drawn on a same half-plane $\mathcal{B}_i$ are either nested, or sit next to each other. This is described by the *nesting relation* $\sqsubset^G \subseteq E^G \times E^G$: given two distinct edges $e_1, e_2 \in E^G$, we write $e_2 \sqsubset^G e_1$ if (1) $x_{e_2} \leq x_{e_1} < y_{e_1} \leq y_{e_2}$ and, (2) either e_1 is a spine edge, or e_1 and e_2 are drawn on the same page. We write $e_2 \sqsubset\!\cdot^G e_1$ whenever $e_2 \sqsubset^G e_1$ and there is no edge e_3 such that $e_2 \sqsubset^G e_3 \sqsubset^G e_1$. By definition, if $e_2 \sqsubset^G e_1$ then e_2 is not a spine edge. The *nesting level* $\mathrm{nl}^G(e)$ of a page edge $e \in E_P^G$ is the cardinality of $\{e' \in E_P^G \mid e' \sqsubset^G e\}$. The nesting level of a spine edge is not defined.

The notion of *facets* in a half-plane is similar to that of faces in the planar case.

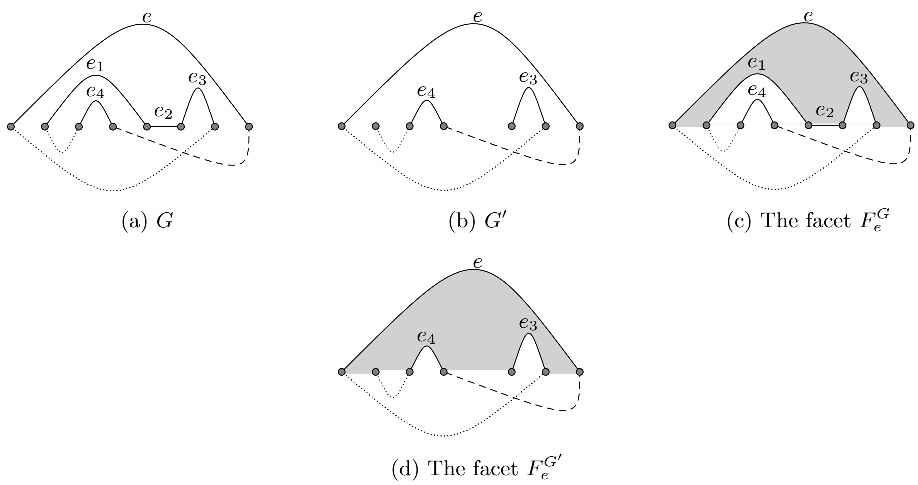
Definition 3. *Let G be an embedding in $\mathcal{B}$. A* facet *of G on page i is a bounded maximal connected component of $\mathcal{B}_i - G$. The* facet *of a page edge e, denoted by F_e^G (or F_e when no confusion arises), is the unique facet of G whose boundary includes e and all edges e' such that $e \sqsubset\!\cdot^G e'$.*

Observe that any facet of G is the facet F_e^G of a page edge e. We extend the nesting level to facets: $\mathrm{nl}^G(F_e) = \mathrm{nl}^G(e)$.

Example 4. Fig. 1-(a) depicts a 3-book embedding G. Vertices are sorted out along an horizontal line, which acts as the spine ℓ. Spine edges are drawn with horizontal straight lines. Page edges are drawn with filled, dotted or dashed arcs. Two arcs drawn with the same style are in the same page. The page where filled (resp. dotted, dashed) edges are drawn is called the filled (resp. dotted, dashed) page.

In G, we have $e_1 \sqsubset\!\!\cdot\; e_4$, $e \sqsubset\!\!\cdot\; e_1$, $e \sqsubset\!\!\cdot\; e_2$ and $e \sqsubset\!\!\cdot\; e_3$. So, $\mathrm{nl}(e) = 0$, $\mathrm{nl}(e_1) = \mathrm{nl}(e_3) = 1$ and $\mathrm{nl}(e_4) = 2$. The nesting level of e_2 is not defined because it is a spine edge. Fig. 1-(b) depicts the subembedding obtained from G by erasing e_1, e_2 and their common vertex. In this subembbeding, the nesting level of e_4 is 1.

In Fig. 1-(c), we have coloured in grey the facet F_e^G. Observe that its boundary includes the edges e, e_1, e_2 and e_3, plus some pieces of the spine (here the pieces between the two leftmost vertices and between the two rightmost vertices).



(a) G (b) G' (c) The facet F_e^G

(d) The facet $F_e^{G'}$

Fig. 1. Examples of a 3-book embedding G (a) and of a subembedding G' of G (b). Two facets of e are depicted in grey, one w.r.t. G (c), and the other w.r.t. G' (d).

Example 4 highlights the fact that the boundary of a facet generally contains points of the spine. Hence, facets are generally neither open nor closed. We define the *frontier* as the part of the boundary that intersects the embedding.

Definition 5. *Let $G = (V, E)$ be an embedding in $\mathcal{B}$. Let S be a subset of $\mathcal{B}$. The* frontier *of S w.r.t. G, denoted by $\mathrm{fr}^G(S)$, is defined by $\mathrm{fr}^G(S) = \mathrm{bd}(S) \cap \bigcup_{e \in E} e$, where $\mathrm{bd}(S)$ refers to the (topological) boundary of S.*

Fontiers can be easily computed from facets.

Lemma 6. *Let G be a book embedding, e be a page edge and S be a set of facets of G. Then*

$$\mathrm{fr}^G(F_e) = e \cup \bigcup_{e \sqsubset\!\cdot^G e'} e' \quad \textit{and} \quad \mathrm{fr}^G(\bigcup_{F \in S} F) = \bigcup_{F \in S} \mathrm{fr}^G(F).$$

When no confusion arises, we omit the superscript G from the above definitions.

3 An outerplanar-like measure for book embeddings

For planar embeddings, outerplanarity is usually defined by deleting external vertices. An alternative measure can be defined by deleting all external edges rather than vertices. These two measures coincide when considering planar embeddings of degree 3 (see [6, Sec. 11.1]). For higher degrees, it is well known that every k-outerplanar graph is a minor of a k-outerplanar graph with degree at most 3.

In this section, we define an outerplanar-like measure for book embeddings, called the *outeredge* measure. It corresponds to peeling a book embedding by the "external" edges. The alternative measure where we peel "external" vertices rather than edges is discussed in Sec. 7. Naturally, we first define in Sec. 3.2 what "external" means in our setting. Our definition uses the concept of regions that we introduce hereafter.

From now on, we fix a p-book $\mathcal{B}$ with p half-planes $\mathcal{B}_1, \ldots, \mathcal{B}_p$.

3.1 Regions

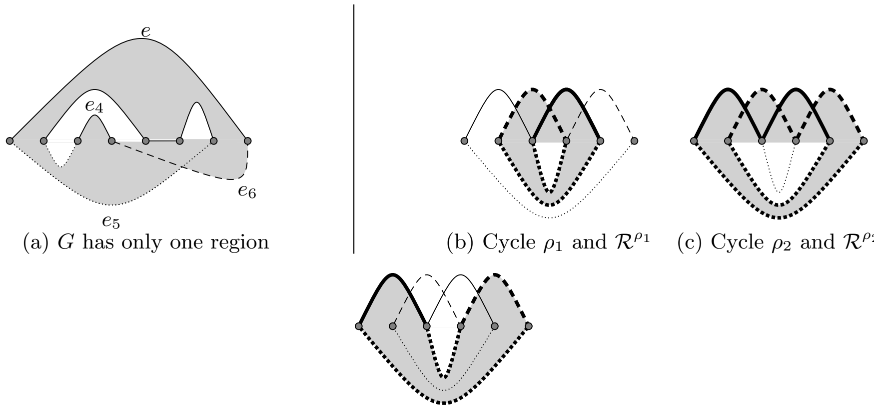
Consider a planar embedding G. A curve ρ of G (namely, the representation of a cycle of G) splits the plane into two maximal connected sets of points with ρ as the boundary. Exactly one of the two is bounded. We denote it by $\mathcal{R}^\rho$. A point x of the plane is internal (with respect to the embedding G) if there is a curve ρ of G such that x belongs to $\mathcal{R}^\rho$.

Following this approach, we define the internal/external points of a book embedding G. However, unlike the planar case, $\mathcal{B} - G$ generally consists of one maximal connected set (consider for instance a curve drawn on two pages of a 3-book). We overcome this problem by building from a cycle ρ of G a particular (non-maximal) bounded connected set $\mathcal{R}^\rho$ of $\mathcal{B}$, called region, that has ρ as frontier. The region $\mathcal{R}^\rho$ is built from facets of the subembedding ρ of G by selecting, on every page, one facet out of two with respect to $\sqsubset^\rho$.

Definition 7. *Let G be a book embedding. The* region $\mathcal{R}^\rho$ of a simple cycle ρ *of G is the union of facets* of the subembedding ρ *having an even nesting level in ρ.*

Observe that a region $\mathcal{R}^\rho$ of an embedding G relies on the drawing of ρ only. Thus, the region $\mathcal{R}^\rho$ is the same for any two embeddings sharing the same subembedding ρ.

Example 8. Examples of regions are coloured in grey in Fig. 2-(a-d). The 3-book embedding represented in Fig. 2-(a) is a simple cycle. It has a single region that is the union of all the facets of even nesting level, namely F_e, F_{e_4}, F_{e_5} and F_{e_6}. The 3-book embedding of Fig. 2-(b-d) reveals three cycles ρ_1, ρ_2 and ρ_3 depicted by bold lines. They involve three regions $\mathcal{R}^{\rho_1}$, $\mathcal{R}^{\rho_2}$ and $\mathcal{R}^{\rho_3}$ of G built from the facets of the subembeddings ρ_1, ρ_2 and ρ_3, respectively. This is why two vertices and one edge of $G - \rho_3$ appear inside $\mathcal{R}^{\rho_3}$ in Fig. 2-(c). Also, observe that the boundary of $\mathcal{R}^{\rho_1}$ includes a piece of the spine (between the third and the fourth vertices). Thus, a region is generally neither open nor closed.



(a) G has only one region (b) Cycle ρ_1 and $\mathcal{R}^{\rho_1}$ (c) Cycle ρ_2 and $\mathcal{R}^{\rho_2}$

(d) Cycle ρ_3 and $\mathcal{R}^{\rho_3}$

Fig. 2. Illustration of regions. Each region (painted in grey) is built from a cycle of the graphs by picked out a suitable set of facets in the subbembedding representing this cycle (depicted in bold). This is carried out regardless of the vertices and the edges that are not in the cycle.

Interestingly, regions have almost the same topological properties as faces. First, like faces, a region $\mathcal{R}^\rho$ defines a set of points whose the frontier is the curve ρ.

Proposition 9. *Let G be a book embedding and ρ be a simple cycle of G. Then,* $\mathrm{fr}(\mathcal{R}^\rho) = \rho$.

Proof. Without loss of generality, we can assume that G consists of ρ only, because $\mathcal{R}^\rho$ is built from the subembedding ρ without regarding the other edges of G. In this case, $\mathrm{fr}(\mathcal{R}^\rho) \subseteq \rho$. Let us prove that $e \in \mathrm{fr}(\mathcal{R}^\rho)$ for all edges e of the cycle ρ. We distinguish two cases, according to whether e is a page edge or a spine edge.

Suppose that e is a page edge. If $\mathrm{nl}(e)$ is even, then $F_e \subseteq \mathcal{R}^\rho$ by Definition 7, and $e \in \mathrm{fr}(F_e) \subseteq \mathrm{fr}(\mathcal{R}^\rho)$ by Lemma 6. If $\mathrm{nl}(e)$ is odd, then there exists a unique e' of ρ such that $e' \sqsubset\!\!\cdot\, e$ and $\mathrm{nl}(e') = \mathrm{nl}(e) - 1$ is even. Then $F_{e'} \subseteq \mathcal{R}^\rho$ by Definition 7, and $e \in \mathrm{fr}(F_{e'}) \subseteq \mathrm{fr}(\mathcal{R}^\rho)$ by Lemma 6.

Suppose that e is a spine edge. Since ρ is a simple cycle, then $\rho - e$ is a simple path between the endpoints of e. This path necessarily uses an odd number of page edges $e' \in \rho$ such that $e' \sqsubset e$. Therefore, there is a page with an odd number of page edges e' on this page such that $e' \sqsubset e$. Among these edges, we select the one that also satisfies $e'' \sqsubset\!\!\cdot\, e$ and we denote it by e''. Clearly, e'' has an even nesting level, so $F_{e''} \subseteq \mathcal{R}^\rho$ by Definition 7, and $e \in \mathrm{fr}(F_{e''}) \subseteq \mathrm{fr}(\mathcal{R}^\rho)$ by Lemma 6.

Note that $\mathcal{R}^\rho$ is generally not the unique union of facets of ρ whose frontier is ρ (see Fig. 2). However if we consider embeddings that do not use all pages (this is not a restriction, since we can add an empty page to $\mathcal{B}$ without modifying $\mathcal{R}^\rho$), then $\mathcal{R}^\rho$ is the unique set of facets such that the frontier of $\mathcal{B} - (\mathcal{R}^\rho \cup \rho)$ is ρ. Besides, regions are bounded and connected, although not maximal. (see A for technical details). This suggests that Definition 7, which may seem somewhat arbitrary, is relevant since it confers on regions the same properties as faces in planar graphs.

3.2 Exteriority and outeredge book embeddings

Definition 10. *Let G be a book embedding.*

- *A point x of $\mathcal{B}$ is* internal *in G if there is a simple cycle ρ in G such that $x \in \mathcal{R}^\rho$.*
- *A vertex v of G is* internal *in G if v is an internal point in G.*
- *An edge e of G is* internal *in G if some point of e° is internal (we recall that e° denotes the set of points of e minus its extremities).*
- *A point, a vertex, an edge or a face that is not internal in G is called* external *in G.*

Note that if an edge e is internal, then every point of e° is also internal.

Remark 11. Consider the book embedding in Fig. 3. This embedding has three cycles : $\rho_1 = 12361$, $\rho_2 = 12541$ and $\rho_3 = 1452361$. The vertices 4 and 5, and the edge $(2, 5)$ are internal because they are in region $\mathcal{R}^{\rho_1}$. The vertex 3 and the edge $(2, 3)$ are internal because they are in region $\mathcal{R}^{\rho_2}$. The edge $(1, 2)$ is internal because it is in region $\mathcal{R}^{\rho_3}$.

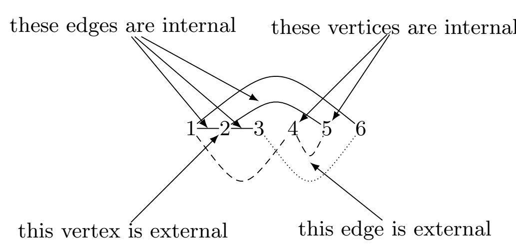


Fig. 3. Example of internal and external edges and vertices in a 3-book embedding. Unlike planar embedding, the endpoints of an external edge can be internal, and all the incident edges of an external vertex can be internal.

Remark 11 and Fig. 3 show that a vertex can be external (here 2) whereas all its incident edges are internal. Also, an edge (here $(4, 5)$) can be external whereas its two endpoints are internal. This differs from planar graphs. It is not hard to prove that such a situation is possible for page edges only.

Proposition 12. *Let $e = (x_1, x_2)$ be a spine edge of a book embedding G. If x_1 or x_2 is internal, then e is internal.*

Proof. If x_i is internal, then there are a cycle ρ of the embedding G and a facet $F \in \mathcal{R}^\rho$ such that $x_i \in F$. Moreover x_i cannot be a point of ρ. It follows that the spine edge e cannot be part of ρ. Therefore, $e^\circ \subset F$, namely e is internal.

We can now formally define the outeredge measure.

Definition 13. *Let G be a book embedding. G is* (1-)outeredge *if all its edges are external. It is k*-outeredge *($k > 1$) if deleting* all *the external edges gives a $(k-1)$-outeredge embedding.*

For technical reasons, we often iteratively delete *some* external edges rather than *all* external edges.

Definition 14. *Let G be a book embedding. A* k-peeling *of G is a sequence $\mathfrak{p} = (G_0, \ldots, G_k)$ of $k+1$ book (sub)embeddings such that $G_0 = G$, G_k is a maximal spanning forest of G, and for all $i \in \{0, \ldots, k-1\}$, G_{i+1} is obtained from G_i by removing* some *external edges of G_i.*

In particular, any forest has a 0-peeling. A k-peeling $\mathfrak{p} = (G_0, \ldots, G_k)$ of a graph $G = (V, E)$ can be fully defined by a labelling: $l^\mathfrak{p} : E \to \{0, \ldots, k\}$ such that: for all $i \in \{0, \ldots, k-1\}$, $l^\mathfrak{p}(e) = i$ if e is an external edge removed from G_i to G_{i+1}; and $l^\mathfrak{p}(e) = k$ if $e \in G_k$.

Clearly, any book embedding that admits a k-peeling is $k+1$-outeredge. The converse property is not trivial because we have to determine which edges to peel to get a maximal spanning forest. As a consequence, the proof requires showing that peeling some external edges rather than all edges does not affect the remaining regions (and then the set of external edges) at each step of the peeling process.

Proposition 15. *Any book embedding that admits a k-peeling is $(k+1)$-outeredge. Any k-outeredge book embedding admits a k-peeling.*

Proof. The first proposition is obvious since a forest is 1-outeredge. We prove the second one by using an induction on k. Let G be a k-book embedding.

Base case. If G is 1-outeredge then all its edges are external and $\mathfrak{p} = (G, T)$ is a 1-peeling of G for any maximal spanning forest T of G ($G = T$ if G is already a forest).

Inductive step. Suppose that $G = (V, E)$ is k-outeredge. Then the subembedding $G'_1 = (V, E'_1)$ obtained by removing from G all the external edges is $(k-1)$-outeredge. By induction hypothesis, G'_1 admits a $(k-1)$-peeling $\mathfrak{p}' = (G'_1, \ldots, G'_k)$. Let M be a maximal set of edges of $E - E'_1$ that can be added to G'_k without creating cycles. We prove that $\mathfrak{p} = (G_0, G_1, \ldots, G_k)$ is a k-peeling of G, where $G_0 = G$ and each G_i ($i \neq 0$) is the graph G'_i enriched by all the edges of M.

By construction, G_k is a maximal spanning forest of G. So, it remains to prove that each G_{i+1} ($0 \leq i \leq k-1$) is obtained by deleting some external edges from G_i. This is clearly true for $i = 0$. For $i \geq 1$, we prove that the set of cycles of G_i consists of cycles of G'_i only: thus regions in G_i and regions in G'_i are the same, which implies that external edges in G'_i are still external in G_i. To do this, we show by contradiction that no cycle of G_i involves an edge of M. Suppose that such a cycle ρ exists. Then ρ can be written as a concatenation of paths $(x_0, y_0)\cdot\zeta_0\cdot(x_1, y_1)\cdots\zeta_{m-1}\cdot(x_m, y_m)$ where $y_m = x_0$ and, for all j, $(x_j, y_j) \in M$ and ζ_j is a path of G'_i from y_j to x_{j+1}. Since G'_k is a maximal spanning forest of G'_1 and since G'_i is a subgraph of G'_1, there exists also a path ζ'_j from y_j to x_{j+1} in G'_k. Then,$(x_0, y_0)\cdot\zeta'_0\cdot(x_1, y_1)\cdots\zeta'_{m-1}\cdot(x_m, y_m)$ is a cycle (not necessary simple) of G_k. This contradicts the fact that G_k is a forest.

4 Treewidth of *k*-outeredge covered-spine book embeddings

The class of k-outeredge p-book embeddings does not have bounded treewidth (that is, for all $t \geq 0$, there is a k-outeredge p-book embedding G such that $\text{tw}(G) > t$). Indeed, every grid $n \times n$ has treewidth in $\Theta(n)$ [6, Lemma 88]. However, it admits a 2-outeredge 3-book embedding (see Fig. 4). In this section, we introduce the class of covered-spine embeddings and show its treewidth can be bounded by a function of the outeredge measure.

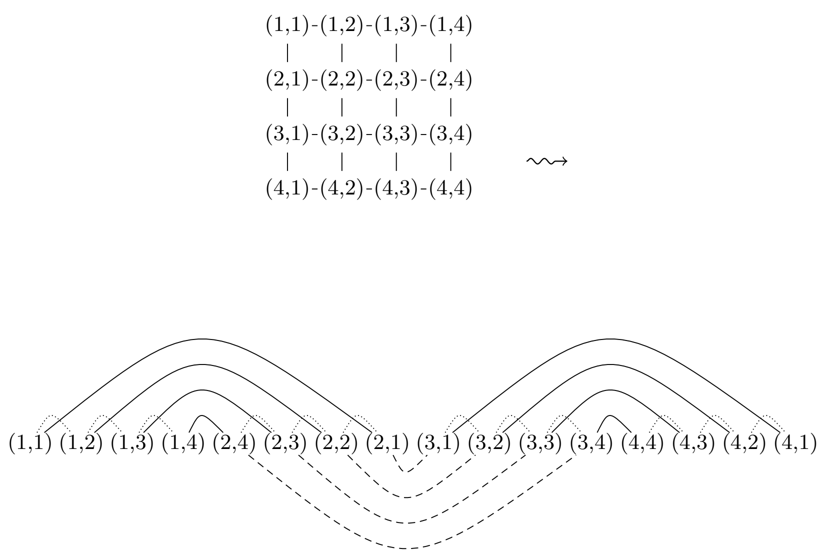


Fig. 4. Embedding a 4×4 grid into a 3-book embedding. All the dotted edges are external. Removing them turns the embedding into a forest. Then the embedding is 2-outeredge.

Definition 16. *A p-book embedding is* covered-spine *if for every* internal *point x of the spine, x belongs to the embedding (namely, x is a vertex, or there is a spine edge e such that $x \in e^\circ$).*

The covered-spine property is a property of the embedding. It does not imply that every pair of consecutive vertices on the spine is an edge of the graph. On the other hand, an embedding G is not necessarily covered-spine whenever every pair of consecutive vertices is an edge of G. Such an embedding is depicted in Fig. 4. The latter is not covered-spine because the points of the spine between the vertices $(2, 4)$ and $(2, 3)$ are internal and they do not belong to the embedding. For instance, the book embeddings of Fig. 2, 3, 4, 5-(b) and 6-(b) are not covered-spine whereas those of Fig. 5-(a) and 6-(a) are.

We can now express the main result of this paper.

Theorem 17. *Let G be a k-outeredge covered-spine p-book embedding of degree $d \geq 3$. Then*

$$\begin{array}{ll} \mathrm{tw}(G) \leq \frac{3d}{2}(p-1)^k & \text{if } p \geq 3, \\ \mathrm{tw}(G) \leq dk+1 & \text{if } p = 2. \end{array}$$

Roadmap of the proof The rest of this section is dedicated to the proof of Theorem 17. Similarly to [6], it is based on the concept of fundamental cycles that we introduce now. Let $G = (V, E)$ be a graph and $T = (V, E_T)$ be a maximal spanning forest of G. For any $E' \subseteq E$, we denote by $T \oplus E'$ the graph $(V, E_T \cup E')$ and we simply write $T \oplus e$ when E' is a singleton $\{e\}$. Let e be an edge of $E - E_T$. The *fundamental cycle* of e with respect to T, denoted by $\rho(e)$, is the unique cycle in $T \oplus e$. The *edge remember number* $\mathrm{er}(G, T)$ of G relative to T is $\max_{\varepsilon \in E_T} |\{e \in E - E_T \mid \varepsilon \in \rho(e)\}|$.

Theorem 18 shows how edge remember number and treewidth are strongly interrelated.

Theorem 18 ([6]). *Let T be a maximal spanning forest of a graph G. Then,* $\mathrm{tw}(G) \leq \frac{1}{2}\deg(G) \cdot \mathrm{er}(G, T) + 1$.

Let $G = (V, E)$ be a k-outeredge covered-spine p-book embedding and $\mathfrak{p} = (G_0, \ldots, G_k)$ be a peeling of G with associated labelling l. By definition of the peeling, G_k is a maximal spanning forest of G. For readability, we denote it by T. Note that T is also a maximal spanning forest of each G_i. We write $G - T$ to refer to the graph or the subembedding obtained from G by removing all edges that belong to T (we also remove the remaining vertices without incident edges). Moreover, E_i denotes the set of external edges removed from G_i to G_{i+1}. Trivially, $T \oplus \bigcup_{i<k} E_i$ equals to G_0, and every edge in E_i is external in $T \oplus E_i$ $(0 \leq i < k)$. It follows that each G_i is a subembedding of G_{i-1} containing at least all internal edges of G_{i-1}.

We aim to compute the edge remember number $\mathrm{er}(G, T)$, and then to apply Theorem 18. For this purpose, we prove that for all edges ε of T, all fundamental cycles in T that pass through ε can be arranged into a p-ary tree of height k (each node corresponding to a fundamental cycle), called the *edge remember tree* of ε. Thus, the edge remember number $\mathrm{er}(G,T)$ simply matches to the maximum number of nodes in such trees.

The proof of Theorem 17 is organised as follows. In Sec. 4.1, we discuss good properties of covered-spine book embeddings. One of them provides a simple characterisation of internal edges that is very useful. However, it is important to stress that for any $e \in G - T$, $T \oplus e$ is generally not covered-spine. Then this characterisation cannot be applied anywhere along the proof, which adds some technical difficulties. In Sec. 4.2, we introduce the notion of *fundamental facets* from which we define the parent function of edge remember trees (see Definition 26). At last, we compute the number of nodes of these trees in Sec. 4.3, which leads to the result.

4.1 Good properties of covered-spine book embeddings

Interestingly, the frontier of an internal facet (namely a facet that consists of internal points only) of a covered-spine book embedding is always a cycle.

Proposition 19. *Let F^G be a facet of a covered-spine book embedding G. If there is a simple cycle ρ of G such that $F^G \subseteq \mathcal{R}^\rho$ then $\mathrm{fr}(F^G)$ is a simple cycle of G.*

Proof. By contradiction, suppose that $F^G \subseteq \mathcal{R}^\rho$ and $\mathrm{fr}(F^G)$ is not a cycle. Then there is a point $h \in \mathrm{bd}(F^G)$ of the spine that does not belong to G. Moreover, $h \in F^G \subseteq \mathcal{R}^\rho$, that is, h is internal. This contradicts that G is covered-spine.

It results a new characterisation of internal edges in terms of facets rather than regions.

Proposition 20. *Let G be a covered-spine book embedding. An edge e is internal if, and only if, there exist two distinct facets F_1 and F_2 in G such that $\mathrm{fr}(F_1)$ and $\mathrm{fr}(F_2)$ are simple cycles, and $e \in \mathrm{fr}(F_1) \cap \mathrm{fr}(F_2)$.*

Proof. If statement: Suppose that $F_1 = F^G_{e_1}$ for some page i and some page edge $e_1 \in E^G_i$ distinct from e (if $e = e_1$ then swap F_1 and F_2). Since $\mathrm{fr}(F_1)$ and $\mathrm{fr}(F_2)$ are simple cycles of G that go through e (by hypothesis), there is a simple cycle ρ in G that uses a subset of edges of $\mathrm{fr}(F_1) \cup \mathrm{fr}(F_2)$ only, and that contains e_1 but not e (since: either $F_2 = F^G_e$ and e and e_1 are drawn in the same page; or e is a spine edge and F_1 and F_2 are on distinct pages). Clearly, $nl^\rho(e_1) = 0$ and then $F^\rho_{e_1} \subseteq \mathcal{R}^\rho$. In addition e° is included in $F^\rho_{e_1}$, and hence e is internal.

Only if statement: Suppose now that e is internal: there is a simple cycle ρ and a facet $F^\rho_{e_1}$ in the (sub)embedding ρ such that $e^\circ \subseteq F^\rho_{e_1} \subseteq \mathcal{R}^\rho$. We distinguish two cases depending on whether e is a page edge or not. If e is a page edge, we have necessarily that $F^G_e \subseteq F^\rho_{e_1} \subseteq \mathcal{R}^\rho$ because ρ is a subembedding of G, In addition, there is $e' \sqsubset\!\cdot^G e$ (e' can be e_1) such that $F^G_{e'} \subseteq F^\rho_{e_1} \subseteq \mathcal{R}^\rho$. Clearly, $e \in \mathrm{fr}(F^G_e) \cap \mathrm{fr}(F^G_{e'})$ and from Proposition 19, $\mathrm{fr}(F^G_e)$ and $\mathrm{fr}(F^G_{e'})$ are simple cycles.

Otherwise e is a spine edge not used in ρ, and since $e^\circ \subseteq F^\rho_{e_1} \subseteq \mathcal{R}^\rho$, we have that $e_1 \sqsubset\!\cdot^{\rho\oplus e} e$ ($\rho \oplus e$ is the embedding that consists of ρ plus the edge e) and that

$\mathrm{nl}^\rho(e_1)$ is even. This means that the number of page edges $e' \sqsubset^{\rho\oplus e} e$ on the same page i as e_1 is odd. However the total number (on both pages) of page edges e' such that $e' \sqsubset^{\rho\oplus e} e$ is even because ρ is a cycle. Therefore there is another page edge e_2 on some page $j \neq i$ such that $e_2 \sqsubset\!\cdot^{\rho\oplus e} e$ and $\mathrm{nl}^\rho(e_2)$ is even. This means that $e^\circ \subset F^\rho_{e_2} \subseteq \mathcal{R}^\rho$. Since ρ is a subembedding of G, there exist also in G two facets with e in their frontier, one included in $F^\rho_{e_1}$ and the other included in $F^\rho_{e_2}$. By Proposition 19, the frontiers of these two facets are cycles.

Proposition 20 can be slightly refined: a page edge of a covered-spine embedding cannot be on the frontier of two distinct facets of the same region.

Proposition 21. *Let ρ be a simple cycle of a covered-spine book embedding G and e be an edge of ρ. If there exist two distinct facets F_1 and F_2 of G such that $F_1, F_2 \subseteq \mathcal{R}^\rho$ and $e \in \mathrm{fr}(F_1) \cap \mathrm{fr}(F_2)$, then e is an internal spine edge of G.*

Proof. Suppose that e is a page edge. Then the facets F_1 and F_2 are actually F^G_e and $F^G_{e'}$ with $e' \sqsubset\!\cdot^G e$. Since ρ is a subgraph of G and since $e \in \rho$, we have that $F^G_e \subseteq F^\rho_e$ and $F^G_{e'} \subseteq F^\rho_{e''}$ for some $e'' \sqsubset\!\cdot^\rho e$. By definition of $\mathcal{R}^\rho$, F^ρ_e and $F^\rho_{e''}$ cannot both be included in $\mathcal{R}^\rho$. Therefore, F^G_e and $F^G_{e'}$ cannot both be included in $\mathcal{R}^\rho$ as well, which contradicts the hypothesis. So e is a spine edge, and from Proposition 19, $\mathrm{fr}(F_1)$ and $\mathrm{fr}(F_2)$ are simple cycles of G. Then Proposition 20 implies that e is an internal spine edge.

At last, the covered-spine property is preserved by peelings.

Lemma 22. *If G is covered-spine and $\mathfrak{p} = (G_0, G_1, \dots, G_k)$ is a k-peeling of G, then each G_i is covered-spine.*

Proof. By induction on i. For $i = 0$, G_0 is covered-spine by hypothesis. Suppose that G_{i-1} is covered-spine and let x be an internal point of the spine in G_i. Then x is also an internal point in G_{i-1} and, moreover, it belongs to a spine edge e of G_{i-1}. Using Proposition 12 or Definition 10 depending on whether x is an endpoint of e or not, we get that e is internal in G_{i-1}. Consequently, e is still an edge of G_i.

4.2 Fundamental facets and edge remember trees

From now on, we fix a k-outeredge covered-spine p-book embedding G and a peeling $\mathfrak{p} = (G_0, \dots, G_k)$ of G with associated labelling l. We set $T = G_k$. We first introduce the fundamental facets of the edges of $G - T$ relative to the maximal spanning forest T. Thereafter, we sort fundamental cycles out in edge remember trees in order to count them. The construction of these trees relies on fundamental facets.

Consider an edge e of $G - T$ and the associated subembedding $T \oplus e$ (see Fig. 5 for an example). The unique cycle of $T \oplus e$ is the fundamental cycle $\rho(e)$, which defines the region $\mathcal{R}^{\rho(e)}$. Since $\rho(e)$ is a subembedding of $T \oplus e$, some facets of $T \oplus e$ are included in $\mathcal{R}^{\rho(e)}$. Only one of them, called the fundamental facet of e, has e in its frontier.

Lemma 23. *Let $e \in G - T$. Exactly one facet of $T \oplus e$ is included in $\mathcal{R}^{\rho(e)}$ and includes e in its frontier.*

Proof. We prove that such a facet exists. By Proposition 9, e is included in the frontier of $\mathcal{R}^{\rho(e)}$. From Definition 7 and Lemma 6, there is a facet F of the embedding $\rho(e)$ included in $\mathcal{R}^{\rho(e)}$ whose frontier includes e. Since $\rho(e)$ is a subembedding of $T \oplus e$, there is also a facet F' of $T \oplus e$ included in F (and then in $\mathcal{R}^{\rho(e)}$) whose frontier includes e.

Next we prove the uniqueness. Let $i = l(e)$. By construction, e is external in G_i. By hypothesis, G is covered-spine, so G_i is too (by Lemma 22). The cycle $\rho(e)$ exists in G_i because it exists in the subembedding $T \oplus e$ of G_i. Consequently, Proposition 21 implies that there exists at most one facet F in G_i included in $\mathcal{R}^{\rho(e)}$ whose frontier includes e. Since $T \oplus e$ is a subembedding of G_i, there is also at most one facet in $T \oplus e$ included in $\mathcal{R}^{\rho(e)}$ whose frontier includes e.

Definition 24. *Let $e \in G - T$. The* fundamental facet *$F(e)$ is the unique facet of $T \oplus e$ that is included in $\mathcal{R}^{\rho(e)}$ and whose the frontier includes e.*

Examples of fundamental facets can be found in Fig. 5 and 6. It is worthwhile to recall here that, although G is covered-spine, $T \oplus e$ is generally not. For instance, the subembedding $T \oplus e$ of Fig. 5-(b) is not covered-spine because there is no spine edge between the third and the fourth vertices (from the left). In Fig. 5, we can see that every edge of $G - T$ included in the region $\mathcal{R}^{\rho(e)}$ carries a label greater than that of e. This fact can be generalized in Lemma 25.

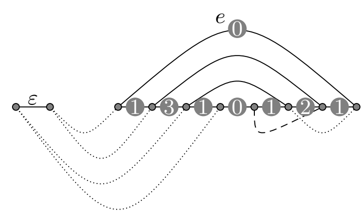
Lemma 25. *Let $e_1, e_2 \in G - T$. If $e_1^\circ \subset F(e_2)$ then $l(e_1) > l(e_2)$.*

Proof. Let $i = l(e_2)$. By construction, e_2 was removed from G_i to G_{i+1}. This means that the cycle $\rho(e_2)$ exists in G_i. By Definition 24, $F(e_2) \subseteq \mathcal{R}^{\rho(e_2)}$. Thus e_1 is internal in G_i and $l(e_1) > i$.

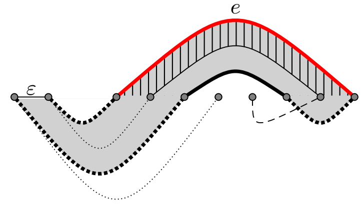
We now explain how we sort fundamental cycles out in edge remember trees. For each edge ε of T, we define a tree whose nodes are the edges e in $G - T$ whose fundamental cycle $\rho(e)$ uses ε. The parent of a node e in the tree is another node whose corresponding edge is included in the fundamental facet $F(e)$.

Definition 26. *Let ε be an edge of T. An* edge remember tree *of ε is a tree structure $(N_\varepsilon \cup \{\varepsilon\}, f_\varepsilon)$ where $N_\varepsilon = \{e \in G - T \mid \varepsilon \subseteq \rho(e)\}$ and such that the parent function $f_\varepsilon : N_\varepsilon \to N_\varepsilon \cup \{\varepsilon\}$ fulfils, for all $e \in N_\varepsilon$: if $I_e = \{e_1 \in N_\varepsilon \mid e_1^\circ \subset F(e)\} \neq \emptyset$, then $f_\varepsilon(e)$ is a minimal (with respect to l) element of I_e; otherwise, $f_\varepsilon(e) = \varepsilon$.*

Generally, there are several edge remember trees for ε. Since all edge remember trees of ε have the same number of nodes, we can choose any of these trees. From now on, we fix an edge ε of T and an edge remember tree of ε denoted by $ERT_\varepsilon = (N_\varepsilon \cup \{\varepsilon\}, f_\varepsilon)$.



(a) A 4-peeling $\mathfrak{p}$ of G



(b) Fundamental facet $F(e)$ with respect to T^p

Fig. 5. Example of a fundamental facet. Figure (a) depicts a 4-peeling $\mathfrak{p}$ of an embedding G with its associated labelling. Removing all labelled edges results the maximal spanning forest $T^{\mathfrak{p}}$. Figure (b) depicts the subembedding $T^{\mathfrak{p}} \oplus e$ of G. The region $\mathcal{R}^{\rho(e)}$ of its unique cycle $\rho(e)$ is painted in grey. The striped part of $\mathcal{R}^{\rho(e)}$ is the fundamental facet $F(e)$ with respect to $\mathfrak{p}$. It is the facet of $T^{\mathfrak{p}} \oplus e$ that is included in $\mathcal{R}^{\rho(e)}$ and that has e in its frontier.

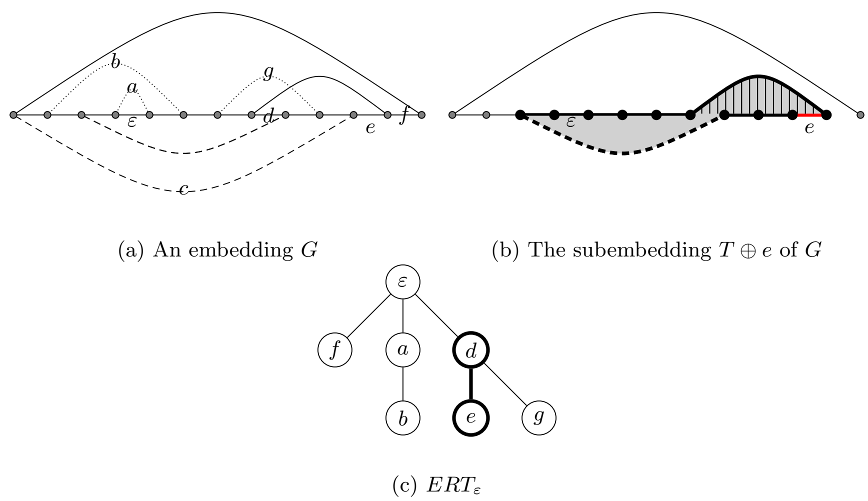
Example 27. Consider Fig. 6. The spanning tree T is obtained from the embedding G of Fig. 6-(a) by the peeling defined by the labelling $l : l(x) = 0$ for all edges $x \in \{b, c, e, f, g\}$, and $l(x) = 1$ for all edges $x \in \{a, d\}$. Figure 6-(c) depicts an edge remember tree ERT_ε of ε associated to the peeling. Unlike edge c, the edges a, b, d, e, f, g are nodes of N_ε because their fundamental cycles go through ε. The node d is the parent of e in ERT_ε because d is a minimal edge (with respect to l) of ERT_ε included in the fundamental facet $F(e)$. The nodes a, d and f are the children of the root ε because their fundamental facets include no node of ERT_ε.

4.3 Number of nodes of an edge remember tree and treewidth

As mention before, the edge remember number (and then the treewidth of G through Theorem 18) actually matches the maximal number of nodes in the edge remember trees. We show that ERT_ε is a p-ary tree of height at most k.

Lemma 28. *ERT_ε is a tree of height at most k.*

Proof. By definition, for any $e_1 \in N_\varepsilon$, if $f_\varepsilon(e_1) = e_2 \neq \varepsilon$ then $e_2^\circ \subset F(e_1)$, and therefore $l(e_2) > l(e_1)$ by Lemma 25. It follows there is no cycle in ERT_ε. Since (1) every edge of N_ε distinct from ε has an image by f_ε, (2) the range of labels is $\{0, \ldots, k\}$, and (3) only the root ε has label k (only edges of T are labelled by k), ERT_ε is a tree of height at most k.



(a) An embedding G

(b) The subembedding $T \oplus e$ of G

(c) ERT_ε

Fig. 6. Construction of an edge remember tree. (b) depicts the subembedding $T \oplus e$ of the embedding G of (a). The fundamental cycle of e (in bold) goes through ε. The region $\mathcal{R}^{\rho(e)}$ is painted in grey and the fundamental facet $F(e)$ is the striped part of $\mathcal{R}^\rho$. The latter includes the edge d, whose fundamental cycle goes through ε as well. It follows that e is a child of d in the edge remember tree ERT_ε (c).

We prove that every node e of ERT_ε has at most p children by distinguishing two cases depending on whether e is ε or not. Suppose that $e \neq \varepsilon$. By definition, if two nodes e_1 and e_2 have the same parent e, then $e \subset F(e_1)$ and $e \subset F(e_2)$. This means that the fundamental facets $F(e_1)$ and $F(e_2)$ intersect one another. If moreover $F(e_1)$ and $F(e_2)$ are on the same page, then Lemma 29 implies that either e_1 or e_2 is a page edge. This fact will be used in Lemma 30 to precisely compute the number of children of e.

Lemma 29. *Let e_1, e_2 be two distinct nodes of N_ε. Suppose that $F(e_1)$ and $F(e_2)$ are on the same page, and that $F(e_1) \cap F(e_2) \neq \emptyset$. Then one, and only one, of the next statements holds: either $e_1^\circ \subset F(e_2)$ and e_1 is a page edge; or $e_2^\circ \subset F(e_1)$ and e_2 is a page edge.*

Proof. Suppose that $F(e_1)$ and $F(e_2)$ are on the half-plane $\mathcal{B}_i$. If $e_1^\circ \subset F(e_2)$ and $e_2^\circ \subset F(e_1)$, then Lemma 25 implies that $l(e_1) > l(e_2)$ and $l(e_2) > l(e_1)$, which is impossible. Suppose now that $e_1^\circ \not\subset F(e_2)$. Then $F(e_2) \cup e_2$ is a connected set of points of $\mathcal{B}_i - (T \oplus e_1)$. Since $F(e_1) \cap F(e_2) \neq \emptyset$ and $F(e_1)$ is a facet of $T \oplus e_1$ (that is, a maximal connected set of points of $\mathcal{B}_i - (T \oplus e_1)$), we have $e_2^\circ \subset F(e_1)$.

Finally we prove by contradiction that e_2 is a page edge. Suppose that e_2 is a spine edge. Then $F(e_2)$ is a facet $F_e^{T \oplus e_2}$ in $T \oplus e_2$ where e is some page edge of T. Now there are two cases: Either e_1 is a page edge, but in this case $e \sqsubset^G e_1$ because $F(e_1) \cap F(e_2) \neq \emptyset$; Or e_1 is a spine edge, but in this case

$F(e_2) = F_e^{T \oplus e_2} = (F_e^{T \oplus e_1} - e_2^\circ) \cup e_1^\circ$ because $F(e_1) \cap F(e_2) \neq \emptyset$. In both cases, this contradicts the fact that $e_1^\circ \not\subset F(e_2)$.

Lemma 30. *Let $e \in N_\varepsilon$. Then e has at most $p-1$ children in ERT_ε.*

Proof. We prove that if e has at least two children, then the fundamental facet of e and those of its children are pairwise on distinct pages. Note that $e \neq \varepsilon$ because $e \in N_\varepsilon$. Let $e_1, e_2 \in N_\varepsilon$ be such that $f_\varepsilon(e_1) = f_\varepsilon(e_2) = e$. From Definition 26, we establish that:

Fact 1. e is a minimal edge of N_ε (with respect to the label) such that $e^\circ \subset F(e_1) \cap F(e_2)$.

Fact 2. $F(e_1)$ and $F(e_2)$ are on distinct pages: otherwise, from Lemma 29, $e_1^\circ \subset F(e_2)$ or $e_2^\circ \subset F(e_1)$, and using Lemma 25, we get $l(e) > l(e_2) > l(e_1)$ or $l(e) > l(e_1) > l(e_2)$ which contradicts the minimality of e.

Now, suppose that $F(e)$ and $F(e_1)$ are on the same page. By Fact 1 and 2, e is necessarily on the spine. Since in addition $e^\circ \subset F(e_1)$, we conclude that $F(e) \cap F(e_1) \neq \emptyset$. Hence we get a contradiction to Lemma 29.

Remark 31. Suppose that $e \in N_\varepsilon$ is a page edge. The proof of Lemma 30 shows that two children of e are on distinct pages. Since a page edge cannot be included in two facets located on different pages, e has at most one child.

Suppose now that $e = \varepsilon$. The contraposition of Lemma 32 states that the frontier of the fundamental facet of any child of ε includes ε. This will be useful to bound the number of children of ε in Lemma 33.

Lemma 32. *Let $e \in N_\varepsilon$. If $\varepsilon \notin \mathrm{fr}^{T \oplus e}(F(e))$, then there is an edge $e_1 \in N_\varepsilon$ with $e_1^\circ \subset F(e)$.*

Proof. By definition of fundamental facets, $F(e) \subseteq \mathcal{R}^{\rho(e)}$ is a set of internal points. Since G is covered-spine, every spine point $h \in F(e)$ is internal, and then belongs to some spine edge e_h of G. We denote by H the set of such spine edges e_h. Necessarily, every $e_h \in H$ does not belong to T (because, on one hand, the fact that h is a point of $F(e)$ implies that ${e_h}^\circ \subseteq F(e)$, and on the other hand, $F(e)$ includes no edge of $T \oplus e$). So, $\rho(e_h)$ is well defined.

We prove by contradiction that there is at least one spine edge e_h of H such that $\rho(e_h)$ goes through ε, that is, such that e_h belongs to N_ε. Suppose that it is not the case. Then the path that consists of the frontier of $F(e)$ in $T \oplus e$, plus the paths $\rho(e_h) - e_h$ for all $e_h \in H$ form together a fundamental cycle for e in $T \oplus e$ that does not use ε (when $H = \emptyset$, the fundamental cycle is simply $\mathrm{fr}(F(e))$, which does not use ε by hypothesis). So we have two fundamental cycles for e in $T \oplus e$ (the first is $\rho(e)$ which goes through ε), which is impossible.

Lemma 33. *The root ε has at most p children in ERT_ε.*

Proof. Suppose that ε has $p+1$ children $e_0, \dots, e_p$ in ERT_ε. Then by Lemma 32, ε is on the frontier of $F(e_i)$ for all $i \in \{0, \dots, p\}$. Since $p \geq 2$ (a book has at least two pages), there are at least three such facets. It follows that ε must be on the spine. Since in addition there are only p pages, two of these fundamental facets, saying $F(e_0)$ and $F(e_1)$ are on the same page and intersect one another. Then Lemma 29 yields that $e_0^\circ \subset F(e_1)$ or $e_1^\circ \subset F(e_0)$, which contradicts the fact that e_0 and e_1 are both children of ε (see Definition 26).

Thus, any edge remember tree is a p-ary tree of height at most k. Since the edge remember number is closely related to the number of nodes of edge remember trees, we can prove our main result (Theorem 17): the treewidth of a k-outeredge covered-spine p-book embedding G of degree $d \geq 3$ is bounded by $\frac{3d}{2}(p-1)^k$ if $p \geq 3$, and by $dk+1$ if $p = 2$.

Proof of Theorem 17. By Proposition 15, G admits a k-peeling $\mathfrak{p}$. We denote by T the maximal spanning forest associated to $\mathfrak{p}$. Clearly, the size of the biggest edge remember tree ERT_ε, where ε ranges over edges of T, precisely corresponds to the edge remember number of G, plus 1 (the root is not a fundamental cycle). From Lemmas 28, 30 and 33, we get that the number of nodes in ERT_ε is bounded by

$$\begin{array}{ll} 1 + \frac{p}{p-2}((p-1)^k - 1) \leq 3(p-1)^k & \text{if } p \geq 3, \\ 2k+1 & \text{if } p = 2. \end{array}$$

Then the result immediately follows from Theorem 18. ∎

5 Treewidth of book embeddings with high degrees

Theorem 17 states that the treewidth of a k-outeredge covered-spine p-book embedding G of degree $d \geq 3$ is bounded by $\frac{3d}{2}(p-1)^k$ when $p \geq 3$. This upper bound depends on the degree of the graph. For graphs with high degree (compared to p), we can improve this bound using the connection between treewidth and minors of Proposition 1. It is known that any graph G is the minor of a 3-degree graph G' that admits a 3-book embedding. This holds because every graph is a minor of a graph with degree at most 3 and, moreover, every graph has a 3-book subdivision [26]. Contrary to Proposition 34, this result does not guarantee that G' has a covered-spine embedding. In addition, it does not take into account the outeredgeness at all.

Proposition 34. *Every graph that admits a k-outeredge covered-spine p-book embedding is the minor of a graph with degree at most 3 that admits a $(k+1)$-outeredge covered-spine p-book embedding.*

Consequently, applying Theorem 17 together with Proposition 1 immediately yields a new bound that no longer depends on the degree. This new bound is better than the previous one when the degree is greater than $3(p-1)$.

Theorem 35. *Let G be a k-outeredge covered-spine p-book embedding. Then*

$$\begin{array}{ll} \mathrm{tw}(G) \leq \frac{9}{2}(p-1)^{k+1} & \textit{if } p \geq 3, \\ \mathrm{tw}(G) \leq 3(k+1)+1 & \textit{if } p = 2. \end{array}$$

The rest of this section is devoted to the proof of Proposition 34. We first describe a procedure M that builds from the embedding G a book embedding $M(G)$ of degree 3. Then, we show that the graph G is a minor of the graph $M(G)$, and that the embedding $M(G)$ is covered-spine and $(k+1)$-outeredge whenever the embedding G is covered-spine and k-outeredge.

The procedure M Let us consider a book embedding $G_0 = (V_0, E_0)$ given in input. Procedure M proceeds as follows. While there exists a vertex v of degree $d > 3$ in G_i (with i initialized to -1) do (see Fig. 7):

Step 1. Increase i by 1 and set $G_{i+1} := G_i$. Let $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ be all vertices of G_{i+1} adjacent to v. We suppose that $x_n < \ldots < x_1 < v < y_1 < \ldots < y_m$ (with respect to the order of the spine ℓ). Since $d > 3$, there are at least two edges adjacent to v that go to the same direction (on the left or on the right). We set $z_1 = x_1$ and $z_2 = x_2$ if these two edges go to the left, otherwise $z_1 = y_1$ and $z_2 = y_2$.

Step 2. Erase (z_1, v) and (z_2, v) from G_{i+1}; Insert a new vertex $\bar{v}$ on the spine between v and the first neighbour of v on the same side as z_1, and add a new spine edge $\bar{e} = (v, \bar{v})$.

Step 3. For each $i \in \{1, 2\}$, there are two cases : If (z_i, v) was a spine edge of G_i, then draw a new spine edge $(z_i, \bar{v})$ in G_{i+1}; If $e_i = (z_i, v)$ was a page edge of G_i, then draw in G_{i+1} a new page edge $\bar{e}_i = (z_i, \bar{v})$ on the same page as e_i in such a way that, for all edges e present in both G_{i+1} and G_i, $e \sqsubset\!\!\cdot^{G_{i+1}} \bar{e}_i$ whenever $e \sqsubset\!\!\cdot^{G_i} e_i$ and $\bar{e}_i \sqsubset\!\!\cdot^{G_{i+1}} e$ whenever $e_i \sqsubset\!\!\cdot^{G_i} e$.

Clearly the procedure ends. The output $M(G_0)$ is the last book-embedding built by the procedure.

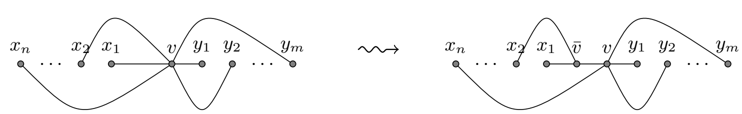


Fig. 7. Reducing the degree of a vertex v while keeping the covered-spine property and the outeredgeness.

We suppose that Procedure M ends after m iterations and that $(G_i = (V_i, E_i))_{i \leq m}$ are the graphs built along Procedure M. From now on, we fix $0 \leq i < m$. According to Procedure M, for some vertex $v \in V_i$ of degree strictly higher than 3, we have $V_{i+1} = V_i \cup \{\bar{v}\}$ and $E_{i+1} = (E_i - \{(z_1, v), (z_2, v)\}) \cup \{(z_1, \bar{v}), (z_2, \bar{v}), \bar{e}\}$. By construction, G_i results from the edge contraction of $(v, \bar{v})$ in G_{i+1}.

Lemma 36. *G_i is a minor of G_{i+1}.*

We show that, if G_i is covered-spine and k-outeredge then G_{i+1} is too. This relies on how regions of G_i are connected to regions of G_{i+1}. We start with some technical definitions and lemmas. Let σ be a surjective map from V_{i+1} to V_i defined by $\sigma(x) = x$ if $x \neq \bar{v}$; otherwise $\sigma(\bar{v}) = v$. We extend it to a bijective map from $E_{i+1} - \{\bar{e}\}$ to E_i: $\sigma(x, y) = (\sigma(x), \sigma(y))$.

Fact 37 *For all vertices $x, y \in V_{i+1}$, $x \leq y$ iff $\sigma(x) \leq \sigma(y)$.*
For all edges $e_1, e_2 \in E_{i+1} - \{\bar{e}\}$, $e_1 \sqsubset^{G_{i+1}} e_2$ iff $\sigma(e_1) \sqsubset^{G_i} \sigma(e_2)$.

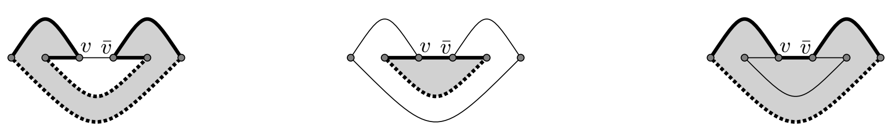
We denote by Φ_{i+1} the set of all simple cycles ρ in G_{i+1} such that, if v and $\bar{v}$ are two vertices of ρ, then ρ goes through $\bar{e} = (v, \bar{v})$. We can extend σ to a map from Φ_{i+1} to cycles of G_i as follows: for all $\rho = (v_1, \ldots, v_r, v_1) \in \Phi_{i+1}$,

$$\begin{cases} \sigma(\rho) = (\sigma(v_1), \ldots, \sigma(v_i), \sigma(v_{i+2}), \ldots, \sigma(v_r), \sigma(v_1)) \\ \qquad\qquad\qquad\qquad \text{if } (v_i, v_{i+1}) = \bar{e} \text{ for some } i \in [1, r-1] \;; \\ \sigma(\rho) = (\sigma(v_1), \ldots, \sigma(v_r), \sigma(v_1)) \text{ otherwise.} \end{cases}$$

Lemma 38. *Let x be a point of the spine not between v and $\bar{v}$. If ρ is a cycle of Φ_{i+1}, then $x \in \mathcal{R}^\rho$ iff $x \in \mathcal{R}^{\sigma(\rho)}$.*

Proof. We prove the only-if-implication only, the proof of the if-implication being very similar. Let us suppose that $x \in \mathcal{R}^\rho$. Then there is a page edge e_1 of ρ such that $x \in F^\rho_{e_1}$ and $\mathrm{nl}^\rho(e_1)$ is even (by definition of $\mathcal{R}^\rho$). Since e_1 is a page edge, $e_1 \neq \bar{e}$. So, $\sigma(e_1)$ is well defined and is part of $\sigma(\rho)$. Moreover, x belongs to the facet $F^{\sigma(\rho)}_{\sigma(e_1)}$ because of Fact 37 and x is not between v and $\bar{v}$. Fact 37 also implies that one iteration of Steps 1-7 preserves the nesting level of edges (with respect to a cycle of Φ_{i+1}), that is, $\mathrm{nl}^\rho(e) = \mathrm{nl}^{\sigma(\rho)}(\sigma(e))$ for any page edge e of ρ. So $\mathrm{nl}^\rho(e_1) = \mathrm{nl}^{\sigma(\rho)}(\sigma(e_1))$, which implies that $\mathrm{nl}^{\sigma(\rho)}(\sigma(e_1))$ is even. Consequently, $x \in F^{\sigma(\rho)}_{\sigma(e_1)} \subseteq \mathcal{R}^{\sigma(\rho)}$.

Next we define two functions π_1 and π_2 that map every cycle of G_{i+1} onto Φ_{i+1} as follows. If ρ belongs to Φ_{i+1}, then $\pi_1(\rho) = \pi_2(\rho) = \rho$. Otherwise, ρ can be rewritten as $(v, v_1, \ldots, v_i, z_1, \bar{v}, z_2, v_{i+1}, \ldots, v_r, v)$, and we set $\pi_1(\rho) = (v, v_1, \ldots, v_i, z_1, \bar{v}, v)$ and $\pi_2(\rho) = (\bar{v}, z_2, v_{i+1}, \ldots, v_r, v, \bar{v})$ (z_1 and z_2 are the vertices of Step 1 in Procedure M). Fig. 8 illustrates these projections.



(a) ρ and its region (b) $\pi_1(\rho)$ and its region (b) $\pi_2(\rho)$ and its region

Fig. 8. Projection of a cycle onto cycles of Φ by π_1 and π_2.

Lemma 39. *Let x be a point of the spine not between v and $\bar{v}$. If ρ is a simple cycle of G_{i+1}, then $x \in \mathcal{R}^{\rho}$ implies that either $x \in \mathcal{R}^{\sigma(\pi_1(\rho))}$ or $x \in \mathcal{R}^{\sigma(\pi_2(\rho))}$.*

Proof. If $x \in \mathcal{R}^{\rho}$ then $x \in F^{\rho}_{e_0}$ for some page edge $e_0 \in \rho$ such that $\mathrm{nl}^{\rho}(e_0)$ is even. Clearly, e_0 is either an edge of $\pi_1(\rho)$ or an edge of $\pi_2(\rho)$. Suppose that e_0 is an edge of $\pi_1(\rho)$ (the case where e_0 is an edge of $\pi_2(\rho)$ is symmetric). Since $\pi_1(\rho)$ is a subembedding of ρ, $F^{\rho}_{e_0} \subseteq F^{\pi_1(\rho)}_{e_0}$. We distinguish two cases depending on the parity of $\mathrm{nl}^{\pi_1(\rho)}(e_0)$. If $\mathrm{nl}^{\pi_1(\rho)}(e_0)$ is even, then $x \in F^{\pi_1(\rho)}_{e_0} \subseteq \mathcal{R}^{\pi_1(\rho)}$ by Definition 7, and the result holds by Lemma 38.

Otherwise $\mathrm{nl}^{\pi_1(\rho)}(e_0)$ is odd. For $\rho' \in \{\rho, \pi_1(\rho), \pi_2(\rho)\}$, we denote by $\#_{\rho'}(e)$ the number of page edges e' in ρ' such that $e' \sqsubset^{\rho'} e_0$. Using this definition, we get that $\#_{\rho}(e_0)$ is even and $\#_{\pi_1(\rho)}(e_0)$ is odd. Since $\rho = \pi_1(\rho) \cup \pi_2(\rho) - \{\bar{e}\}$ and $\pi_1(\rho) \cap \pi_2(\rho) = \{\bar{e}\}$, we have $\#_{\rho}(e_0) = \#_{\pi_1(\rho)}(e_0) + \#_{\pi_2(\rho)}(e_0)$. Consequently, $\#_{\pi_2(\rho)}(e_0)$ is odd and, since $\pi_2(\rho)$ is a cycle, there exists a page edge e' in $\pi_2(\rho)$ such that $e' \sqsubset^{\rho} e_0$. We choose such an edge with the greater nesting level and denoted by e_1. Then $\mathrm{nl}^{\pi_2(\rho)}(e_1)$ is even and $x \in F^{\rho}_{e_0} \subseteq F^{\pi_2(\rho)}_{e_1} \subseteq \mathcal{R}^{\pi_2(\rho)}$. The result holds by, Lemma 38.

Lemma 40. *If G_i is covered-spine, then G_{i+1} is too.*

Proof. Let x be an internal spine point of G_{i+1}. If x is between v and $\bar{v}$, then x belongs to $\bar{e}$. Otherwise, x is also an internal spine point of G_i by Lemma 39. So x belongs to a spine edge of G_i because we have supposed that G_i is covered-spine. Then, by construction of G_{i+1}, x still belongs to an edge of G_{i+1}.

From now on, we suppose that G_i is covered-spine. Then G_{i+1} is also covered-spine by Lemma 40. Consequently, we can use Proposition 20 to determine whether an edge is external or not. Let *peel* be the function that maps a book embedding to a new book embedding obtained by removing all the external edges. We write $peel^j$ the map obtained by j compositions of *peel*.

Lemma 41. *Let $j \geq 0$. If there is a cycle in $peel^j(G_{i+1})$ then there is a cycle in $peel^j(G_i)$.*

Proof. First we prove by induction on j the next property:

(P) : Let $e \neq \bar{e}$ be an edge of G_{i+1} and j be a positive integer. If e is an edge of $peel^j(G_{i+1})$ then $\sigma(e)$ is an edge of $peel^j(G_i)$.

For $j = 0$, we have $peel^0(G_{i+1}) = G_{i+1}$ and $peel^0(G_i) = G_i$. Then the result trivially holds since σ is a bijection from $E_{i+1} - \{\bar{e}\}$ to E_i.

Inductive case ($j > 0$). Let $e \in E_{i+1} - \{\bar{e}\}$ be an edge in $peel^j(G_{i+1})$. Since e is in $peel^j(G_{i+1})$, e is an internal edge in $peel^{j-1}(G_{i+1})$. By Proposition 20, there exist two facets F_1 and F_2 such that $e \in \mathrm{fr}(F_1) \cap \mathrm{fr}(F_2)$ and, $\mathrm{fr}(F_1)$ and $\mathrm{fr}(F_2)$ are simple cycles of Φ. Let e_1 and e_2 be the page edges such that $F_1 = F^{peel^{j-1}(G_{i+1})}_{e_1}$ and $F_2 = F^{peel^{j-1}(G_{i+1})}_{e_2}$. Using the induction hypothesis and the fact that σ maps cycles of Φ_{i+1} to cycles in G_i, there are two facets $F'_1 = F^{peel^{j-1}(G_i)}_{\sigma(e_1)}$ and $F'_2 = F^{peel^{j-1}(G_i)}_{\sigma(e_2)}$ whose frontiers are simple cycles such that $\sigma(e) \in F'_1 \cap F'_2$.

So by Proposition 20, $\sigma(e)$ is internal in $peel^{j-1}(G_i)$, and is still an edge of $peel^{j}(G_i)$.

As an immediate consequence of (P), if ρ is a cycle of $peel^{j}(G_{i+1})$, then $\sigma(\pi_1(\rho))$ or $\sigma(\pi_2(\rho))$ is a cycle of $peel^{j}(G_i)$.

Finally, we can prove that any graph that admits a k-outeredge covered-spine p-book embedding G is a minor of a graph with degree at most 3 that admits the k-outeredge covered-spine p-book embedding $M(G)$.

Proof of Proposition 34. Let G_0 be a k-outeredge covered-spine book embedding. By Lemma 36, G_0 is a minor of G_m, and by Lemma 40, the embedding G_m is covered-spine. Since G_0 is k-outeredge, $peel^{k+1}(G_0)$ has no cycle (by Proposition 15). By Lemma 41, that is the case for all $peel^{k+1}(G_i)$, $i \in \{1, \ldots, m\}$. In particular, this means that all edges of $peel^{k+1}(G_n)$ are external. So G_m is $(k+1)$-outeredge (by Proposition 15). ■

6 A lower bound for the class of *k*-outeredge covered-spine book embeddings

We denote by $\mathcal{C}_{p,k}$ the class of graphs that admit a k-outeredge covered-spine p-book embedding. We prove that $\mathrm{tw}(\mathcal{C}_{p,k}) = \Omega(2^k)$ for a fixed p. For this purpose, we use results from automata theory and verification areas. Indeed, Theorem 3.11 from [24] states that the emptiness problem for p-stack pushdown automata (p-PDA) is decidable in 2ETime when restricted to particular computation graphs: the class $\mathcal{B}_{p,k}$ of k-phase p-nested words (the number of stacks p is fixed).

Theorem 42. *[24] For $k \in \mathbb{N}$, the emptiness problem for p-PDAs M restricted to $\mathcal{B}_{p,k}$ is decidable in time $|M|^{O(\mathrm{tw}(\mathcal{B}_{p,k}))}$.*

The authors of [24] point out this result matches the 2ETime lower bound for this problem given in [17]. Recall that 2ETime is the class of all decision problems that can be solved by a deterministic Turing machine in time $2^{2^{dn}}$ for some constant d. This implies the following proposition.

Proposition 43. $\mathrm{tw}(\mathcal{B}_{p,k}) = \Omega(2^k)$.

Multi-nested words [23] correspond to multi-pushdown graphs of degree 3 in [14]. Formally, a p-nested word is a graph $N = (V, E)$ where V is a finite set of vertices, and E is a disjoint union of sets of edges L, $E_1, \ldots, E_p$ such that

- $L \subseteq V \times V$ is a non-reflexive successor edge relation such that L^* is a linear ordering $<_L$ on the vertices of V;
- for every $0 < j \leq p$, E_j is a nesting matching: for all $u, u', v, v' \in V$ and $1 \leq j' \leq p$: if $E_j(u, v)$ and $E_{j'}(u', v')$, then u, v, u', v' are all different; if $E_j(u, v)$ and $E_j(u', v')$ and $u <_L u'$, then either $v <_L u'$ or $v >_L v'$ holds.

Intuitively, p-nested words capture the behaviours of runs of p-pushdown automata. A relation E_j corresponds to the matching push-pop relation on stack j. Observe that every vertex of a nested word has at most one edge in both E_j's. This means that a p-pushdown automaton cannot perform two push/pop actions at the same time. In the following definition, a phase corresponds to any sequence of actions on stacks such that all pops in the sequence are on the same stack. A p-nested word N is k-phase if, $\bigcup_{1\leq j\leq p} E_j$ can be partitioned into k sets $Phase_1,\ldots,Phase_k$ such that:

- for all $1 \leq i \leq k$, all edges in $Phase_i$ are in E_j for some $1 \leq j \leq p$;
- for all $1 \leq i < i' \leq k$, if $(v,w) \in Phase_i$ and $(v',w') \in Phase_{i'}$, then $w < w'$.

There is a trivial embedding of a p-nested word N into a p-book: vertices are drawn on the spine from left to right with respect to $<_L$; the successor edges of L are drawn on the spine; for every $1 \leq j \leq k$, the nested edges of E_j are drawn on page j. This embedding is covered-spine. We show that it admits a k-peeling whenever N is k-phase.

Proposition 44. *Any k-phase p-book nested word admits a k-peeling p-book embedding.*

Proof. Let $G_0 = (V, L, \{E_j\}_{0<j\leq p})$ be a k-phase p-book nested word. Without loss of generality, we suppose that $V = \{0,\ldots,n-1\}$ where n is the number of vertices in V, and that $L = \{(x,x+1) \in V^2 \mid 0 \leq x < n-1\}$. We denote by $Phase_1,\ldots,Phase_k$ the partition of $\bigcup_{0<j\leq p} E_j$ into k phases. We consider the trivial embedding of G_0 as described above and denote by p_i the page associated with $Phase_i$. We prove the proposition by induction on k. The base case $k = 1$ is trivial: since every page edge is drawn on the page p_1, G_0 admits a 1-peeling. For the induction step, we fix $k > 1$ and consider G_1 to be the book embedding built from G_0:

- by removing all page edges $e \in Phase_k$; (by definition of k-phase, all these page edges are located on page p_k);
- by removing, for all $e = (x,y) \in Phase_k$, the spine edge $(y-1,y)$. We denote by S the set of spine edges thus removed.

Clearly, $G_1 = G_0 - (Phase_k \cup S)$ is a $(k-1)$-phase p-book embedding. By induction, G_1 admits a $(k-1)$-peeling $\mathfrak{p}' = (G_1,\ldots,G_k)$. We prove that $\mathfrak{p} = (G_0, G_1 \cup Phase_k, \ldots, G_k \cup Phase_k)$ is a k-peeling of G_1 in six steps.

Any spine edge $s \in S$ is external in G_0. By construction, there is at most one facet that includes s in its frontier. Hence, by Proposition 20, and since G_0 is covered-spine, s is external.

$G_1 \cup Phase_k$ is covered-spine. The deletion of external edges preserves the property of being covered-spine, then since G_0 is covered-spine, $G_1 \cup Phase_k = G_0 - S$ is too.

Any edge e of $Phase_k$ is external in $G_1 \cup Phase_k$. Since $e = (x,y)$ is a page edge, only two facets contain e in its frontier, whose one is $F_e^{G_1\cup Phase_k}$. However, the frontier of $F_e^{G_1\cup Phase_k}$ cannot form a cycle because, by construction, the spine edge $(y-1, y)$ missed in $G_1 \cup Phase_k$. So we conclude using Proposition 20 and the fact that $G_1 \cup Phase_k$ is covered-spine that e is external in $G_1 \cup Phase_k$.

Each G_i is covered-spine. Since any edge of $Phase_k$ is external in $G_1 \cup Phase_k$ and $G_1 \cup Phase_k$ is covered-spine, so is G_1. From Lemma 22, each G_i is covered-spine.

Any external edge e in G_i is still external in $G_i \cup Phase_k$. Since G_i is covered-spine, from Proposition 20 there is at most one facet F in G_i whose frontier includes e and forms a cycle. In addition, F is also a facet of $G_i \cup Phase_k$. Any facet of $G_i \cup Phase_k$ that does not exist in G_i is of type $F_{e'}^{G_i \cup Phase_k}$ where $e' = (x', y') \in Phase_k$. However, the frontier of such a facet cannot form a cycle because, by construction, the spine edge $(y'-1, y')$ missed in $G_i \cup Phase_k$. Therefore, we conclude using Proposition 20 that e is still external in $G_i \cup Phase_k$.

$G_k \cup Phase_k$ is a maximal spanning forest of G_0. By definition of $\mathfrak{p}'$, G_k is a maximal spanning forest of G_1. Suppose that $G_k \cup Phase_k$ is not a forest. Then there is a cycle in $G_k \cup Phase_k$ that necessary uses an edge of $Phase_k$. Let x be the smallest node such that there is an edge $e = (x, y)$ of $Phase_k$ used by such a cycle ρ. Consider $\pi = \rho - e$, since by construction, the spine edge $(y-1, y)$ is not an edge of $G_1 \cup Phase_k$, it is not in $G_k \cup Phase_k$ too. Then the spine edge $(y, y+1)$ is part of π. Consequently, in order to join x, π necessary passes through an edge $e' = (x', y')$ on a page $i \neq p_k$ such that $x' < y < y'$. This contradicts the fact that G_0 is k-phase.

To conclude: $G_1 \cup Phase_k$ is obtained from G_0 by removing external edges since edges of S are external in G_0 and $G_1 \cup Phase_k = G_0 - S$; for $i = 1, \ldots k-1$, $G_{i+1} \cup Phase_k$ is obtained from $G_i \cup Phase_k$ by removing external edges only (since any external edge in G_i is still external in $G_i \cup Phase_k$ and $\mathfrak{p}'$ is a $(k-1)$-peeling); $G_k \cup Phase_k$ is a maximal spanning forest of G_0. Then $\mathfrak{p}$ is a k-peeling of G_0.

Proposition 44 together with Proposition 15 entail that $\mathcal{B}_{p,k} \subseteq \mathcal{C}_{p,k+1}$. Applying Proposition 43 leads to a lower bound on the treewidth of $\mathcal{C}_{k,p}$.

Theorem 45. *Let $p \geq 3$ be a fixed integer,* $\mathrm{tw}(\mathcal{C}_{p,k}) = \Omega(2^{k-1})$.

Remark 46. Note that authors of [24] have built a long proof to compute the treewidth of a k-phase nested word (Lemma 3.10 in [24]). Proposition 44 together with Theorem 35 gives another proof. This illustrates how our results can be applied to the field of formal verification.

7 Discussion

Up to now, we have been interested in the outeredge measure (see Definition 13). To end this paper, we discuss an alternative measure where vertices are peeled rather than edges.

Definition 47. *Let G be a book (or planar) embedding. G is* (1-)outervertex *if all its vertices are external. It is k-*outervertex *$(k > 1)$ if deleting all the external vertices (and their adjacent edges) gives a $(k-1)$-outervertex embedding.*

In planar graphs, the k-outervertex measure is usually called k-outerplanarity. It trivially coincides with the k-outeredge measure for graphs of degree 3. We study the relationship between the outeredge measure and the outervertex measure in the setting of book embeddings. We start with a simple result that also holds for the plane. It follows from the observation that, in the plane or a 2-book, a vertex is external as soon as one of its incident edges is external. We recall that this generally fails for book embeddings (c.f. Remark 11).

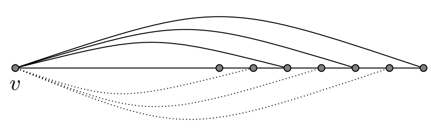
Proposition 48. *A k-outeredge 2-book embedding is $k+1$-outervertex.*

Proof. Let G be a k-outeredge 2-book embedding. We use an induction on k.

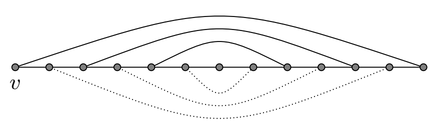
Basis: For $k = 1$, every edge of G is external. Then every endpoint of an edge is external too. So, removing all external vertices deletes all edges of G. The resulting embedding is trivially 1-outervertex. By definition, this means that G is 2-outervertex.

Inductive step ($k > 1$). Let G' be the $(k-1)$-outeredge embedding obtained by removing all external edges. In a 2-book, every endpoint of some external edge of G is external. So, removing external vertices gives a subembedding G" of G'. Then, G'' is at most $(k-1)$-outeredge. By the induction hypothesis, G'' is k-outervertex, which means that G is $(k+1)$-outervertex.

The converse does not hold. Indeed, consider the covered-spine 2-book embedding where the set of vertices is $\{1, 2, \ldots, n\}$, the set of spine edges is $\{(j, j+1) \mid 1 \leq j < n\}$, the set of edges on the first page is $\{(1, j) \mid 2 < j \leq n \text{ and } j \text{ is even}\}$ and the set of edges on the second page is $\{(1, j) \mid 2 < j \leq n \text{ and } j \text{ is odd}\}$ (see Fig. 9-(a) for an example with $n = 8$). This embedding is 1-outervertex and $n/2$-outeredge.



(a) G is 2-outervertex and 4-outeredge



(b) $M(G)$ is 4-outervertex and 4-outeredge

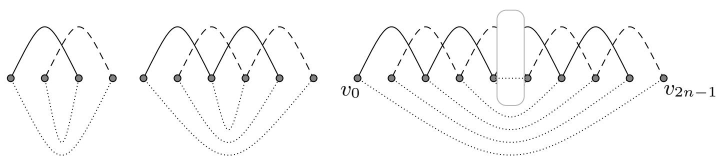
Fig. 9. Procedure M does not preserve the outervertex measure.

Proposition 48 fails as soon as we consider graphs with 3 pages.

Proposition 49. *For all $k > 2$, there is a 3-book embedding of degree 3 that is 2-outeredge and k-outervertex.*

Proof. Let C_n be the 3-book embedding of degree 3 depicted in Figure 10. It consists of $2n$ vertices $0, \dots, 2n-1$ ordered on the line from left to right, filled edges $(2i, 2i+2)_{i \in [0,n-2]}$ drawn on the first page, dashed edges $(2i+1, 2i+3)_{i \in [0,n-2]}$ drawn on the second page, and dotted edges $(i, 2k-1-i)_{i \in [0,n-1]}$ on the third page.

Clearly, filled edges and dashed edges are all external. Then C_n is 2-outeredge. Note that the only external vertices of C_n are the leftmost and the rightmost vertices. So C_n is not 1-outervertex. Removing these vertices (and the incident edges) gives C_{n-1}. By induction, we get that C_n is exactly n-outervertex, C_1 being trivially 1-outervertex.



(a) C_2 (b) C_3 (c) C_n

Fig. 10. A 2-outeredge-bounded class that is not outervertex-bounded. In C_n, filled edges and dashed edges are all external, whereas only vertices v_0 and v_{2n-1} are external.

As already mentioned, Theorem 83 from [6] states that every k-outerplanar embedding has treewidth at most $3k-1$. Such a result fails for k-outervertex p-book embeddings since, for any $n > 0$, the $n \times n$ grid is the minor of a graph that can be embedded in a 1-outervertex 3-book embeddings. For instance consider the embedding of the grid 4×4 of Fig. 4. Each dotted edge $((i,j),(i,j+1))$ can be subdivided once into two edges $((i,j),x_j)$ and $(x_j,(i,j+1))$ where x_j is a new vertex. The new vertices are drawn to the right of the embedding such that $x_{j'} < x_j$ if $j < j'$. Their incident edges are drawn on the dotted page. This embedding is 2-outervertex since removing all the x_j's gives a forest.

In Theorems 17 and 35, we give upper bounds on the treewidth of k-outeredge covered-spine book embeddings. We can ask whether such results exist for k-outervertex covered-spine book embeddings. If we take a closer look at the proof of [6, Theorem 83], we can see that it uses a peeling of the external edges rather than external vertices. In this way, the author reduces the problem to k-outeredge planar embeddings of degree 3. This is possible because: (1) every k-outerplanar embedding is a minor of a k-outerplanar embedding with degree at most 3; (2) if a k-outerplanar embedding has degree at most 3, then removing all its external edges gives a $(k-1)$-outerplanar embedding.

We think that the point (2) still holds for k-outervertex covered-spine book embeddings but the proof seems very technical. Also, we can ask whether every k-outervertex covered-spine book embedding is a minor of a k-outervertex covered-spine book embedding with degree at most 3. That is, does there exist a similar result to Proposition 34? Procedure M presented in Sec. 6 does not work. As a counterexample, the embedding $M(G)$ in Figure 9 becomes 4-outervertex while G is 2-outervertex only. The existence of such a procedure is open. The difficulty is to preserve the covered-spine condition.

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A On topological properties of regions

In this section, we discuss about topological properties of regions as defined in this paper. Let ρ be a book embedding that consists of a simple cycle. We have already mentioned (see Proposition 9) that the frontier of the region $\mathcal{R}^\rho$ is ρ. However, $\mathcal{R}^\rho$ is generally not the unique union of facets satisfying this property, but it is the unique one whose the frontier of its complement is also ρ (see Proposition 50, which actually holds when there exists a page without edge. However, this restriction can always be supposed in this paper without loss of generality). Furthermore, like regions surrounded by a curve in the plane, the region $\mathcal{R}^\rho$ is a bounded connected set (Lemma 51 and Proposition 52). This suggests that our definition of regions, which may seem somewhat arbitrary, is topologicaly relevant in the sense it confers on regions the same properties as faces in planar graphs.

Proposition 50. *Let S be a union of facets of a book embedding that consists of a simple cycle ρ. If $\mathrm{fr}(S) = \rho$ and $\mathrm{fr}(\mathcal{B} - (S \cup \rho)) = \rho$ then $S = \mathcal{R}^\rho$. Moreover, the converse holds whenever a page of the book $\mathcal{B}$ is empty.*

Proof. We recall that $\mathcal{R}^\rho$ is the union of all facets F_e such that $\mathrm{nl}^\rho(e)$ is even. We prove the contrapositive. Let us suppose that $S \neq \mathcal{R}^\rho$. Then, one of the following cases holds:

Case 1. There are page edges e and e' with $e' \sqsubset\!\cdot\, e$ such that F_e and $F_{e'}$ are included in S. Then, $e \cap \mathrm{fr}(\mathcal{B} - (S \cup \rho)) = \emptyset$.

Case 2. There are page edges e and e' with $e' \sqsubset\!\cdot\, e$ such that F_e and $F_{e'}$ are not included in S. In this case, $e \cap \mathrm{fr}(S) = \emptyset$.

Case 3. If cases 1 and 2 fail, then S necessarily collects all facets F_e such that $\mathrm{nl}^\rho(e)$ is odd. In this case, for all edges e with $\mathrm{nl}^\rho(e) = 0$, $e \cap \mathrm{fr}(S) = \emptyset$.

We show that the converse holds when a page $\mathcal{B}_i$ of the book contains no page edge. Clearly, by construction of $\mathcal{R}^\rho$, every page edge e of the embedding is also included in $\mathrm{fr}(\mathcal{B} - (\mathcal{R}^\rho \cup \rho))$. Then, any neighbourhood of a point x of any spine edge e meets $\mathcal{B} - (\mathcal{R}^\rho \cup \rho)$ because it meets the empty page $\mathcal{B}_i$. Also, it trivially meets $\mathcal{R}^\rho \cup \rho$. Consequently, x (and then the points of e) belongs to the frontier of $\mathcal{B} - (\mathcal{R}^\rho \cup \rho)$.

Lemma 51. *Let ρ be a book embedding that consists of a simple cycle. If all page edges have a null nesting level then $\mathcal{R}^\rho$ is a connected set.*

Proof. We prove this property by induction on the number of edges. We first observe that $\mathcal{R}^\rho = \bigcup_{e \in E_P} F_e$ because all page edges have a null nesting level. The basis case of a graph with only two edges is trivial. For the induction case, let us consider x_1 to be the smallest vertex drawn on the spine (w.r.t. the linear order $<$ defined over the points the spine). There are two edges $e_1 = (x_1, x_3)$ and $e_2 = (x_1, x_2)$ such that $x_1 < x_2 < x_3$ and e_1 is not a spine edge. If the path from x_2 to x_3 is fully drawn on the spine, then $\mathcal{R}^\rho = F^\rho_{e_1} \cup F^\rho_{e_2}$ is connected (with the convention that $F_{e_2} = \emptyset$ if e_2 is a spine edge). Else, since ρ is a cycle, there is a page edge $e = (x, y)$ (drawn on a different page than e_1) with $]x, y[\cap]x_2, x_3[\neq \emptyset$ and then F_{e_1} and F_e are connected. Finally, let ρ' be the simple cycle obtained from ρ by removing e_1 and e_2 and replace them by the edge $e' = (x_2, x_3)$ drawn on the same page as e_1. By induction, $\mathcal{R}^{\rho'}$ is a connected set including F^ρ_e. Since $\mathcal{R}^\rho = F^\rho_{e_1} \cup F^\rho_{e_2} \cup (\mathcal{R}^{\rho'} - F^{\rho'}_{e'})$, $\mathcal{R}^\rho$ is also connected.

Proposition 52. *Let ρ be a book embedding that consists of a simple cycle. Then $\mathcal{R}^\rho$ is a bounded connected set .*

Proof. Clearly, $\mathcal{R}^\rho$ is bounded since any facet is bounded. We define the nesting level of a cycle ρ as the sum of the nesting levels of its page edges. We prove by induction on the nesting level of ρ that $\mathcal{R}^\rho$ is connected. The basis case results from Lemma 51.

Suppose that the nesting level of ρ is $w > 0$. Then there is a page p with at least two nested page edges. Let $\alpha_0 = (x_0, y_0)$ and $\alpha_1 = (x_1, y_1)$ be two page edges drawn on page p such that $\alpha_0 \sqsubset^\rho \alpha_1$, $\mathrm{nl}^\rho(\alpha_0) = 0$ and $\mathrm{nl}^\rho(\alpha_1) = 1$. We build from ρ, α_0 and α_1 a new book embedding G as follows (see Fig. 11): remove α_0 and α_1; instead, draw two new page edges on p, e_0 from x_0 to x_1 and e_1 from y_0 to y_1, in such a way that there is no edge e such that $e \sqsubset^G e_0$ or $e \sqsubset^G e_1$ in G (more precisely, we do not add e_0 (resp. e_1) if $x_0 = x_1$ (resp. $y_0 = y_1$)).

By construction, $\mathrm{nl}^G(e_0) = \mathrm{nl}^G(e_1) = 0$. We can easily check that for the other page edges e, drawn on page p

$$\text{either } \mathrm{nl}^G(e) = \mathrm{nl}^\rho(e) \text{ or } \mathrm{nl}^G(e) = \mathrm{nl}^\rho(e) - 2. \qquad (1)$$

Clearly our construction has created at most two cycles ρ_0 and ρ_1 in G. Each of them involves exactly one edge among e_0 and e_1. We suppose that ρ_i

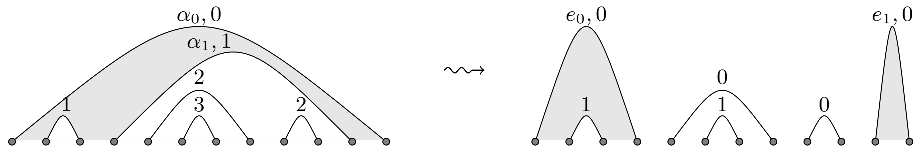

Fig. 11. Construction of a book embedding with a smaller nesting level

(when it exists) goes through e_i, $i < 2$. By Eq. (1), each cycle ρ_i has a smaller nesting level than ρ. Using the inductive hypothesis, each $\mathcal{R}^{\rho_i}$ is connected. Furthermore, $\mathrm{nl}^{\rho_i}(e_i) = 0$ because $\mathrm{nl}^G(e_i) = 0$. It results that $F_{e_i}^{\rho_i} \subseteq \mathcal{R}^{\rho_i}$, and then $F_{e_i}^{\rho_i}$ is connected to $\mathcal{R}^{\rho_i} - F_{e_i}^{\rho_i}$ (2).

Now, according to Eq. (1) and from construction of G, we have $\mathcal{R}^{\rho} = (\mathcal{R}^{\rho_1} - F_{e_1}^{\rho_1}) \cup (\mathcal{R}^{\rho_2} - F_{e_2}^{\rho_2}) \cup F_{\alpha_0}^{\rho}$. By (2), and since we have clearly that $F_{e_i}^{G} \cap \ell \subseteq F_{\alpha_0}^{\rho} \cap \ell$ for $i \in \{0, 1\}$, all points of $\mathcal{R}^{\rho_i} - F_{e_i}^{\rho_i}$ are also connected to $F_{\alpha_0}^{\rho}$. It follows that $\mathcal{R}^{\rho}$ is connected.