The value of network information: Assortative mixing makes the difference

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ABSTRACT

A monopoly sells a network good to a large population of consumers. We explore how the monopoly's profit and the consumer surplus vary with the arrival of public information about the network structure. The analysis reveals that, under homogeneous preferences for the good, degree assortativity ensures that information arrival increases both profit and consumer surplus. In contrast, heterogeneous preferences for the good can create a tension between consumer surplus and profit.

JEL classification: C72 D85

Keywords: Monopoly, Network effects, Network information, Bonacich centrality, Degree assortativity, Assortative mixing

1. Introduction

Consider a consumer estimating his projected consumption of a network good (like a mobile phone, an online game, membership of a social club, etc.). Faced with network effects, he needs to estimate the consumption of his neighbors, which in turn depend on the consumption of their neighbors, and so on. Thus, the consumer really needs to know the complete pattern of interactions among all consumers to get a precise measure of his own consumption needs.

When the consumer has little information about the network structure, he tries to figure out whether his neighbors are more central or peripheral in the network, guessing that centrality is correlated with consumption levels. Now suppose that consumers learn new information about the network structure. Depending on his initial information, a consumer may then become more convinced that his neighbors are central or, conversely, peripheral in the network, with the consequence that some consumers may increase their consumption while others may reduce their consumption. The new information may thus lead to either an increase or a decrease in demand, profit and consumer surplus. This raises the following question: is there a network property ensuring that the arrival of information about the network structure is beneficial to firms and consumers?

We investigate the value of network information in the context of linear price discrimination by a monopolist. The firm sells a divisible good to a large number of consumers organized in a network of local complementarities in consumption

☆ For helpful comments, the authors thank participants in various conferences and seminars. This work was supported by French National Research Agency Grant ANR-17-EURE-0020, and by French National Research Agency Grant ANR-18-CE26-0020-01.

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and positive externalities. Consumer utilities are linear-quadratic, which generates a linear interaction between consumption decisions. For clarity of exposition, our benchmark model assumes homogeneous preferences for the good and undirected networks; this benchmark contains the core analysis of the overall study. Under undirected networks, prices are independent of the network structure, meaning that the firm does not adjust prices with the arrival of information. Hence, both profit and welfare variations are exclusively related to demand effects. Profit is driven by the aggregate consumption level and consumer surplus by the sum of squared consumption levels.

We consider two information sets. Under the initial information set, consumers only know their number of neighbors (i.e., their degree), as well as the degree distribution of the network, and the monopolist observes each consumer's degree. We assume that the degree distribution of potential neighbors is the same for all consumers under the initial information set (Assumption 1). This means that consumers cannot infer any further information about the network structure; in particular, there is no degree correlation between linked consumers. Under the enriched information set, on top of the initial information there is additional and public information about the network structure. For example, this can take the form of information on the distribution of neighbors' degrees conditional on own degree for each consumer, or, in extreme cases, the full network structure.[1] We compare monopoly profit and consumer surplus under the two information sets. We say that a network generates a positive information value for monopoly profit (resp. for consumer surplus) when monopoly profit (resp. consumer surplus) is greater under enriched information. Our main objective is to characterize the types of information on the network structure that drive positive information values.

Under the initial information set, individual consumption is a function of consumer degree, while under the enriched information set it is in general a function of the whole network structure. We then show, and this is our main result (Theorem 1), that both profit and consumer surplus increase with the arrival of network information for all intensities of interaction, if and only if the matrix of interaction representing the enriched information game is degree assortative, i.e. if consumers with similar degrees tend to be linked with each other.[2] The intuitive mechanism at play is that, under degree assortativity, enriched information leads high-degree (resp. low-degree) consumers to increase (resp. decrease) their expectations as to their neighbors' consumption and thus increase (resp. decrease) their own consumption. Now, degree assortativity means that the indirect influence of high-degree consumers on others' behaviors is larger than that of low-degree consumers, which explains the increase in aggregate consumption. And because this increase is driven by high-degree consumers, it also guarantees an increased sum of squared consumption levels. This result is positive, given the well-documented stylized fact that social networks generally exhibit degree assortativity (see Newman (2002), Table I, p. 2, or Serrano et al. (2007)). Its suggests that, for such social networks, providing information about the network structure can create value.

We then incorporate consumers' heterogeneity in individual preferences for the good, assuming that the monopoly fully observes individual characteristics. With heterogeneous preferences, it is not only degree assortativity that plays a role, but also other assortativity coefficients. *Homophily* refers to assortativity by preferences, and indicates that consumers are more likely to be linked to consumers with similar preferences. *Preference-degree assortativity* measures the tendency of high-preference consumers to be linked to high-degree consumers. The positiveness of the three assortativity coefficients guarantees that the arrival of network information will increase profit for all intensities of interaction. For instance, the value of network information can be negative on a network that is degree assortative and homophilic but is preference - degree dis-assortative. Yet these conditions do not necessarily imply increased demand, which explains how consumer surplus can fall when the intensity of interaction is sufficiently high. Hence, in contrast to homogeneous preferences, heterogeneity can create a discrepancy between consumer surplus and profit under assortative mixing.

We extend our results in several directions. First, we explore alternative information structures by relaxing Assumption 1. In particular, we consider initial information sets in which agents have more information than degree distribution alone. Then, extensions are detailed in separate appendices. Appendix B presents directed networks, which mainly differs from undirected networks in that, in this more general setting where prices depend on the network, information arrival induces a price effect. We extend Theorem 1 to directed networks and show that, under degree assortativity of the - symmetric - network of averaged interaction, both profit and consumer surplus increase with information. Appendix C presents two alternative pricing scenarios that confirm the crucial role of assortative mixing: we consider optimal pricing, and we also examine the case where the firm cannot price discriminate by considering a homogeneous fixed price under both the initial and enriched information sets. In Appendix D, we discuss more general utilities with linear best-responses, which is relevant under a deterministic interpretation of the model. All these extensions confirm the key role played by assortative mixing.

Our results have two major policy implications. First, when the network has assortative mixing properties, it is in the firm's interest to make the information available to consumers. Second, a discrepancy between profit and consumer surplus creates a need for regulation, like restricting access to this kind of information, or denying the firm to right to make it available to consumers.

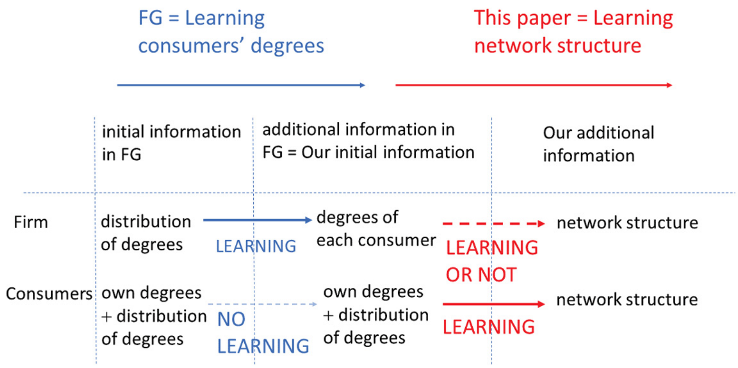
[1] A huge amount of information about consumer relationships is available today, with the rise of digital technologies. Consumers are better informed about the social network structure, thanks to online social networks (like Klout, Facebook, LinkedIn, etc.).

[2] Assortative mixing plays a crucial role in our study. A social network exhibits assortative mixing (Newman (2002)) if there is a positive correlation in the characteristics of people socially connected with each other. In general, the characteristics can be a personal attribute such as age, education, socio-economic status, physical appearance, and religion, or a measure of centrality, e.g., degree, betweenness, Bonacich centrality, etc.

The article is organized as follows. Section 2 discusses the related literature. Section 3 presents the benchmark model under homogeneous preferences for the good and undirected networks. Section 4 examines the sign of the value of information. We explore heterogeneous preferences in Section 5 and Section 6 provides a discussion about consumers' information structures and considers more general utilities. Section 7 concludes. Appendix A contains all proofs, Appendix B presents directed networks and Appendix C explores alternative pricing. Appendix D examines more general utilities. Appendix E presents assortative mixing coefficients in detail.

2. Related literature

Our work contributes to the classical literature on the economics of privacy (see Acquisti et al. (2015) for a recent survey) by highlighting the role of assortative mixing in assessing the value of social network data under markets with network externalities. Such markets are explored by the classical IO literature on network effects initiated by Farrell and Saloner (1985) or Katz and Shapiro (1985). Recently, monopoly pricing under network effects has been studied using two approaches. First, Candogan et al. (2012) and Bloch and Quérou (2013) address this issue under full information. Using a setup that incorporates an explicit network structure, they mainly establish a link between consumption levels and Bonacich centralities.[3] Second, some papers consider incomplete network information. Candogan et al. (2012) compare the case where the monopolist does not know network effects (whereas consumers do know them) to the case of complete information, and quantify the positive impact of network information on the monopolist. Fainmesser and Galeotti (2016) consider incomplete network information in a model where consumers only know the distribution of in-degrees and out-degrees in the network. The authors examine how increased information held by the monopoly about consumers' degrees impacts both monopoly profit and welfare. There are two major differences between FG's model and ours. First, we focus on the arrival of information about the network structure whereas, in FG, the information simply concerns about in- or out-degrees (obviously, knowledge of consumers' degrees alone does not provide a comprehensive description of the network structure). Second, the consumers learn in our setting whereas, in FG, the consumers' information structure is kept fixed. The figure below explains the difference between our paper and Fainmesser and Galeotti (2016) in terms of information structures: our initial information corresponds to their additional information, and our enriched information is about network structure.



It should be stressed that, when the information is about network structure, it is crucial that consumers learn it, otherwise information arrival generates no value. This is obvious in the undirected case where prices are independent of the network structure. However, it is also true in the directed case, because the arrival of information does not affect optimal pricing strategy when demand is kept unchanged. By considering information on network structure, we fill the gap between FG and Candogan et al., and can apply our results to compare the outcomes of the two models.

This paper also builds on the theoretical literature on network games.[4] Ballester et al. (2006) consider network effects in a game of linear-quadratic utilities under complete information, and establish a relation between equilibrium play and Bonacich centrality. An emerging literature, including Bramoullé et al. (2014), follows this pioneering work. Jackson and Yariv (2005, 2007) study diffusion of behavior and equilibrium properties in a network game with incomplete network information. Galeotti et al. (2010) take this a step further, discussing strategic equilibrium in a wide set of network games. Considering utility functions concave in degrees, they compare outcomes from different degree distributions. In contrast, we compare networks with the same joint degree distributions. One contribution of our study is that it incorporates network structure considerations into a model of incomplete network information. Moreover, our approach can be generalized beyond

[3] These papers study linear-in-sum games. Ballester et al. (2006) show that Bonacich centralities play a prominent role in such games. Briefly, this index counts the number of walks to others on the network, where the walk counting is decayed geometrically with walk length through an exogenous decay parameter.

[4] Our work also echoes the literature studying the efficiency of stratification under complementarities. See Becker (1973), Bénabou (1996), Durlauf and Seshadri (2003). Our main departure from this literature lies in introducing a network structure between interacting agents.

monopoly pricing. For example, because our results are driven by demand effects, they extend to the general environment of linear games played on networks. Charness et al. (2014) use experimental economics to test network games with incomplete information. Other theoretical articles on network games with incomplete information include Sundararajan (2008), Kets (2011), De Marti and Zenou (2015), and Lambert et al. (2018). In a setup similar to ours, Jackson (2019) studies the following friendship paradox: on average, people have strictly fewer friends than their friends have. He shows that this may lead to an overestimation of actions for all agents. The friendship paradox does not, however, explain our results, since it is linked to variance in neighbors' degrees whereas our results are driven by degree assortativity, i.e. correlation between neighbors' degrees.

Homophily is widely studied in network economics; see Currarini et al. (2009) for the formation of friendship networks, Golub and Jackson (2012) in the context of learning, Galeotti and Rogers (2013) on strategic immunization incentives, or Cabrales et al. (2014) for risk-sharing networks. However the impact of degree assortativity on economic outcomes has so far received little attention.[5] To our knowledge, this paper is the first to highlight the role of degree assortativity in the context of networked interaction. Interestingly, the literature has identified network formation dynamics that generate degree assortativity. Jackson and Rogers (2007) consider a network formation process mixing random and network-based search. Bramoullé et al. (2012) introduce individual heterogeneity in the framework of Jackson and Rogers (2007), and study how homophily affects network integration. In König et al. (2010), agents form and sever links based on the centrality of their potential partners, and in a context of capacity constraints on the number of links agents can maintain.

3. The model

A single monopolist produces a divisible good at no cost, and charges linear prices in the presence of local network effects among a large population of homogeneous consumers.[6] We consider two information sets, respectively called the initial and the enriched information sets. Under the initial information set, both the firm and consumers know the degree distribution of the network, consumers also know their own degree and the firm knows each consumer's degree. Under the enriched information set, both consumers and the monopoly obtain new and public information (common knowledge) on the network structure, on top of the initial information. The extent of this additional information ranges from small, like the number of links between a given pair of degrees, to large, like the number of links between every pair of degrees or even the full network. The key is that the corresponding game is represented by a matrix of interaction that encodes the relevant information. This requirement is not restrictive. Essentially, consumers update the probability distribution of their neighbors' degrees; what is important is that, under the enriched information set, we can compute the probability that some given link of a given consumer i connects to a consumer with a given degree d.

We present first the benchmark model under complete information, then the game under incomplete information. Importantly, we show a correspondence between the partial information game and a game under complete information where there is an adequate matrix of interaction reflecting the available information. We will use this correspondence extensively to compare outcomes of the two games. Then we present the equilibrium outcomes in unified notation.

Complete information game. We consider a two-stage game, with a set $\mathcal{N} = \{1, 2, \cdots, n\}$ of consumers organized in a social network. In the first period, the monopolist sets prices and in the second period, consumers buy a divisible quantity of the good.

We let the $n \times n$ matrix $\mathbf{G} = [g_{ij}]$, with $g_{ij} \in \mathbb{R}_+$, represent the network of interaction between consumers, with $g_{ij} > 0$ when agent i is influenced by agent j and $g_{ij} = 0$ otherwise. By convention, $g_{ii} = 0$ for all i. We will refer to it as network $\mathbf{G}$. We assume that bilateral interactions are symmetric, i.e. $\mathbf{G}^T = \mathbf{G}$ where superscript ^{T} denotes the transpose operator (Appendix B examines directed networks). Symbols $\mathbf{0}, \mathbf{1}$ represent respectively the n-dimensional vectors of zeros and ones, $\mathbf{I}$ the n-dimensional identity matrix, $\mathbf{d} = \mathbf{G1}$ the profile of consumers' degrees, d_i the degree of consumer i, $g = \mathbf{1}^T\mathbf{G1}$ the sum of degrees in network $\mathbf{G}$. For simplicity and without loss of generality, we exclude isolated consumers from the analysis, i.e. we assume that $d_i \geq 1$ for all i. For all degrees $d \in \{1, \cdots, n-1\}$, we let s_d represent the number of consumers of degree d in network $\mathbf{G}$.

We let $q_i(\mathbf{G}) \in \mathbb{R}_+$ represent the quantity purchased by consumer i for a given network of interactions $\mathbf{G}$, and $\mathbf{q}(\mathbf{G}) = (q_i(\mathbf{G}))_{i \in \mathcal{N}}$ the corresponding vector of consumptions. The monopolist selects a vector $\mathbf{p}(\mathbf{G}) = (p_i(\mathbf{G}))_{i \in \mathcal{N}}$ of prices where $p_i(\mathbf{G}) \geq 0$ represents the linear price charged to consumer i for one unit of the good; for convenience and when there is no possibility of confusion, we may omit reference to network $\mathbf{G}$ in individual variables (Appendix C examines two alternative pricing scenarios). In this context, the monopoly profit on network $\mathbf{G}$ is given by $\Pi(\mathbf{G}) = \mathbf{p}(\mathbf{G})^T\mathbf{q}(\mathbf{G})$. Writing for convenience $\mathbf{q} = (q_i, q_{-i})$, the utility that agent i derives from consuming the quantity q_i of the good on network G is given by:

$$u(q_i, q_{-i}) = q_i - \frac{1}{2}q_i^2 - p_i q_i + \delta \sum_{j \in \mathcal{N}} g_{ij} q_i q_j \tag{1}$$

[5] Degree assortativity has been studied very intensively and prominently in network analyses in physics, in biology and in the social sciences. For instance, degree assortativity is known to play an important role in diffusion processes and has an impact on connectivity properties of large networks.

[6] Introducing a constant and positive marginal production cost would affect outcomes but plays no role in the analysis. We thus normalize the marginal cost to zero.

This linear-quadratic utility specification was introduced by Ballester et al. (2006) and used by Candogan et al. (2012) in the context of monopoly pricing. The last term represents the utility that consumer i derives from neighbors' consumption. We consider $\delta > 0$, which implies positive externalities and local complementarities: incentives to consume increase with neighbors' consumption. The consumer surplus on network $\mathbf{G}$ is given by $\omega(\mathbf{G}) = \sum_{i \in \mathcal{N}} u(q_i, q_{-i})$.

Incomplete information game. The incomplete information setup corresponds to an environment in which consumers are aware of their propensity to interact with other consumers, but they do not know the network structure when taking consumption decisions. To model incomplete information, agents commonly assume that the network is drawn from a random network formation process, i.e. all agents, including the monopolist, have a common prior about the underlying network, and they receive signals. Let $\mathcal{I}$ be the set of information. In the paper we will consider two sets corresponding respectively to the initial and the enriched information sets. Under both information sets, when consumption decisions are made, each consumer i knows her number of neighbors and the degree distribution of her potential neighbors in the network (this probability is updated under information arrival), and the monopoly knows every consumer's degree. In this model, consumers are naturally typed by their degrees. We define by $\mathcal{T} = \{1, 2, \cdots, n-1\}$ the set of consumer types, i.e. degrees. We focus on Bayesian Nash equilibrium with type space $\mathcal{T}$ which are symmetric in degrees, i.e. configurations where all consumers of the same degree t choose the same consumption level, generically called q_t (this assumption is standard in the literature, see Galeotti et al. (2010) and related discussion there-in). We let $P_t^{\mathcal{I}}$ represent the degree distribution of the population of degree-t consumer's potential neighbors (as a proportion of the total population) conditional on information $\mathcal{I}$. Denoting by $\tilde{P}_t^{\mathcal{I}}(t')$ the probability of any neighbor of a degree-t agent having degree t' in network $\mathbf{G}$ conditional on information set $\mathcal{I}$, we have $\tilde{P}_t^{\mathcal{I}}(t') = \frac{t' P_t^{\mathcal{I}}(t')}{E_t[d]}$, where $E_t[\cdot]$ is the expectation associated with $P_t^{\mathcal{I}}$. As explained before, this probability will be updated under information arrival. The expected utility of a degree-t consumer i is given by

$$EU_i(q_i, q_{-i}) = (1 - p_t)q_t - \frac{q_t^2}{2} + \delta q_t t AV_t^{\mathcal{I}}(\mathbf{q})$$

where $AV_t^{\mathcal{I}}(\mathbf{q})$ refers to the expected average consumption level of a degree-t consumer's neighbors under information set $\mathcal{I}$, which is given by $AV_t^{\mathcal{I}}(\mathbf{q}) = \sum_{t' \in \mathcal{T}} \tilde{P}_t^{\mathcal{I}}(t') q_{t'}$.

In order to obtain tractable comparisons between the outcomes of games with distinct information sets, we establish an equivalence between the incomplete information game and a game of complete information with adequate matrix of interaction, similar to that established independently by Fainmesser and Galeotti (2016). To obtain the system of agent interactions for given distribution of neighbors' degrees $\{\tilde{P}_t^{\mathcal{I}}(t')\}_{(t,t') \in \mathcal{T} \times \mathcal{T}}$, we multiply this quantity by t and divide by $s_{t'}$, the number of agents of degree t'. Let $\mathbf{W} = (w_{ij})$, with $w_{ij} = \frac{t \tilde{P}_t^{\mathcal{I}}(t')}{s_{t'}}$ where t is the degree of consumer i and t' the degree of consumer j. This matrix is called the matrix of agent interactions (here termed network $\mathbf{W}$). Consumer i's expected utility becomes $u_i(q_i, q_{-i}) = (1 - p_i)q_i - \frac{q_i^2}{2} + \delta \sum_{j \in \mathcal{N}} w_{ij} q_i q_j$. Networks $\mathbf{W}$ and $\mathbf{G}$ have the same degree distributions, and network $\mathbf{W}$ is usually weighted, even though network $\mathbf{G}$ is binary. The Bayesian Nash equilibrium of the second stage of the incomplete information game played on network $\mathbf{G}$ generates the same consumption profile as the Nash equilibrium of the complete information game played on network $\mathbf{W}$.[7]

We will consider two sets $\mathcal{I}_0$, $\mathcal{I}_1$, corresponding respectively to the initial and the enriched information sets. Under the initial information set $\mathcal{I}_0$, we assume that probability $P_t^{\mathcal{I}_0}(t')$ only depends on t'. Formally, recalling that s_t represents the number of consumers of degree t in network $\mathbf{G}$:

Assumption 1. Under the initial information set $\mathcal{I}_0$, the degree distribution of the population of potential neighbors is the same for all consumers, and is given by $P^{\mathcal{I}_0}(t) = \frac{s_t}{n}$.

This assumption implies that there is no correlation between linked consumers. It requires a large number of consumers, and excludes any network structure whose degree distribution allows consumers to infer more information on the probability distribution of their neighbors (an extreme example is the star network, where the degree distribution fully reveals the network structure). Assumption 1 implies that $\tilde{P}_t^{\mathcal{I}_0}(t') = \frac{t' s_{t'}}{g}$ (given Assumption 1 and noting that $\sum_t t s_t = g$). This ratio is equal to the number of links involving consumers of degree d over the total number of links in the network; i.e. it simply refers to the density of links of degree t-consumers among all links. This assumption is extensively used in Galeotti et al. (2010) and Fainmesser and Galeotti (2016) (we relax Assumption 1 in Section 6; see Propositions 8 and 9). Hence, the expected average consumption level of neighbors is the same for all consumers and given by $AV^{\mathcal{I}_0}(\mathbf{q}) = \sum_{t \in \mathcal{T}} \frac{t s_t}{g} q_t$. The equilibrium outcomes of this incomplete information game can be expressed as functions of the statistics of the degree distribution, and in particular consumption is an affine function of degrees (see equations (8) and (9) in Appendix A). The

[7] This formulation of an equivalent game of complete information is only for technical convenience, for the purposes of our comparative static study. Hence, it is not necessary to alleviate the non-null diagonal entries of matrix $\mathbf{W}$.

system of agent interactions, here called network $\mathbf{H}$ for convenience (i.e., $\mathbf{W}=\mathbf{H}$), is given by $\mathbf{H}=\frac{\mathbf{d}\mathbf{d}^T}{g}$, so that $h_{ij}=\frac{d_i d_j}{g}$ for all i, j, including the diagonal.

Under the enriched information set $\mathcal{I}_1$, agents have access to additional signals, e.g., about assortativity or other structural properties related to the draw of the underlying network; at the extreme, consumers learn the full network structure and play the game of complete information. With this additional information, all consumers update the degree distribution of the population of their potential neighbors. Note that, under the enriched information set, the degree distributions of potential neighbors differ across consumers. We illustrate the representation of enriched information by a matrix of agent interactions with two examples, in decreasing order of network information. We denote by $\mathbf{V}$ the matrix of agent interactions (i.e., $\mathbf{W}=\mathbf{V}$).

Example 1. Under the enriched information set, agents know the number of links between each pair of degrees. We define matrix $\mathbf{\Psi}=[\psi_{tt'}]$ where $\psi_{tt'}$ represents the total number of links between degree-t and degree-t' consumers, where t is the type of consumer i, t' the type of consumer j and s_t the number of consumers of type t. When the network is large, probability $\tilde{P}_t^{\mathcal{I}_1}(t')$ can be approximated by $\frac{\psi_{tt'}}{ts_t}$. In order to obtain the system of agent interactions, we multiply this quantity by t and divide by $s_{t'}$, the number of agents of degree t'. The corresponding matrix of agent interactions therefore satisfies

$$v_{ij}=\frac{\psi_{d_i d_j}}{s_{d_i}s_{d_j}}$$

Example 2. Under the enriched information set, agents know the number of links between a given degree t_0 and all other degrees. That is, agents know all the entries $\psi_{t_0 t'}$ for every degree t' in matrix $\mathbf{\Psi}$ previously defined in Example 1. When the network is large, the matrix of agent interactions encoding this information is written

$$\begin{cases} v_{ij}=\frac{\psi_{d_i,d_j}}{s_{d_i}s_{d_j}} \text{ if } d_i=t_0 \text{ or } d_j=t_0 \\ v_{ij}=\frac{\left(d_i-\frac{\psi_{d_i,t_0}}{s_{d_i}}\right)\left(d_j-\frac{\psi_{d_j,d_0}}{s_{d_j}}\right)}{g-\sum_{t'}\psi_{t_0,t'}} \text{ if } d_i\neq t_0, d_j\neq t_0 \end{cases}$$

Finally, if agents learn the full network under the enriched information set, network $\mathbf{G}$ represents the matrix of the system of agent interactions.

Under the enriched information set, the equilibrium quantities, in particular profit and consumer surplus, can differ from their values under the initial information set. In this paper we compare the equilibrium quantities before/after the arrival of additional information.[8]

Equilibrium outcomes and Bonacich centralities. We let $\mu(\mathbf{W})$ denote the largest eigenvalue of matrix $\mathbf{W}=\{\mathbf{V},\mathbf{H}\}$. We impose the following assumption:

Assumption 2. $\delta\cdot\max(\mu(\mathbf{V}),\mu(\mathbf{H}))<1$

Assumption 2 guarantees that both games, under initial and under enriched information, admit a single and interior solution, by ensuring that optimal consumption levels are finite. For a given price vector, the first-order condition on the demand of consumer i on network $\mathbf{W}\in\{\mathbf{V},\mathbf{H}\}$ is written:

$$q_i^{BR}=1-p_i+\delta\sum_{j\in\mathcal{N}}w_{ij}q_j \tag{2}$$

Optimal profit and consumer surplus can therefore be expressed as functions of the position of agents on the network through Bonacich centralities, $\mathbf{b}(\mathbf{W},\delta)=(\mathbf{I}-\delta\mathbf{W})^{-1}\mathbf{1}$. Assumption 2 guarantees $(\mathbf{I}-\delta\mathbf{W})^{-1}\geq\mathbf{0}$.[9] More generally, for any n-dimensional vector $\mathbf{z}$, the vector $\mathbf{b}_{\mathbf{z}}(\mathbf{W},\delta)=(\mathbf{I}-\delta\mathbf{W})^{-1}\mathbf{z}$ represents the *Bonacich centrality* of network $\mathbf{W}$ weighted by $\mathbf{z}$. For convenience, we will omit reference to parameter δ in centralities, profit, and consumer surplus. We also introduce the Euclidian norm $\|\mathbf{z}\|=\sqrt{\mathbf{z}^T\mathbf{z}}$. Recall that:

8 Note that the Friendship Paradox generally arises in the initial information game, and its prevalence is impacted by information arrival. At the extreme, if the new information is such that there is perfect degree assortativity, the paradox disappears.

9 The inverse matrix can be developed as an infinite sum, so that $\mathbf{b}(\mathbf{W},\delta)=\sum_{k=0}^{+\infty}(\delta\mathbf{W})^k\mathbf{1}$. The quantity $b_i(\mathbf{G},\delta)$ represents the number of walks from agent i to others in network $\mathbf{G}$, where a walk of length k is weighted by factor δ^k. Note that network $\mathbf{H}$ is weighted and has a non-null diagonal; it is still possible to interpret $b_i(\mathbf{H},\delta)$ as a Bonacich centrality if we consider weighted walks.

Proposition 1 *(**Candogan et al. (2012)**). For network* $\mathbf{W} \in \{\mathbf{V}, \mathbf{H}\}$*, prices, consumption levels, optimal monopoly profit, and consumer surplus are written*

$$\begin{cases} \mathbf{p}(\mathbf{W}) = \frac{1}{2}\,\mathbf{1} \\ \mathbf{q}(\mathbf{W}) = \frac{1}{2}\,\mathbf{b}\big(\mathbf{W}\big) \\ \pi(\mathbf{W}) = \frac{1}{4}\,\mathbf{1}^T\mathbf{b}\big(\mathbf{W}\big) \\ \omega(\mathbf{W}) = \frac{1}{8}\,\big\|\mathbf{b}\big(\mathbf{W}\big)\big\|^2 \end{cases}$$

Both profit and consumer surplus depend on the Bonacich centralities. And since network $\mathbf{W}$ is undirected, the price vector is independent of the network structure. Bloch and Quérou (2013) obtain similar characterization when their equilibrium is interior. That is, the price is identical under both information sets (this is not the case for directed networks - see Appendix B). For convenience, vector $\mathbf{y}$ (resp. $\mathbf{x}$) will refer to the consumption profile in the game under initial (resp. enriched) information throughout the article.

4. The value of network information

In this section, we compare monopoly profit and consumer surplus between the two games under initial and enriched information on undirected networks and under homogeneous preferences for the good. Information arrival generates a pure demand effect that modifies both profit and consumer surplus. For clarity, we present intuitions through the benchmark case where enriched information game is the game of complete information. Then we present the main results in the more general case of partial information arrival. However, we first need to define the coefficient of degree assortativity, a cornerstone of the results under homogeneous preferences.

Let coefficient $r_{\mathbf{d}}(\mathbf{V})$ measure the level of (dis)assortative mixing by degree.[10] Degree assortativity (resp. degree disassortativity) holds whenever $r_{\mathbf{d}}(\mathbf{V}) > 0$ (resp. $r_{\mathbf{d}}(\mathbf{V}) < 0$), and indicates that consumers are more (resp. less) likely to be linked to consumers with similar degrees than in a random network with the same degree distribution. As an extreme case, the correlation coefficient of a complete bipartite network, including the star network, takes the value -1. However, in general non-complete bipartite networks can be assortative. By contrast, the Pearson coefficient of the union of two components in which every agent has the same degree with at least two distinct degrees takes the value 1. The following lemma (proved in Appendix E) will be useful for the analysis under homogeneous preferences. It establishes a link between the degree assortativity and the difference between the two matrices of interaction $\mathbf{H}$ and $\mathbf{V}$:

Lemma 1. *Consider an undirected network* $\mathbf{V}$ *with degree vector* $\mathbf{d} = \mathbf{V}\mathbf{1}$ *and* $g = \mathbf{1}^T\mathbf{d}$ *the sum of degrees, let* $\mathbf{H} = \frac{\mathbf{d}\mathbf{d}^T}{g}$ *and let* $\mathbf{\Theta} = \mathbf{V} - \mathbf{H}$*. Then,*

$$r_{\mathbf{d}}(\mathbf{V}) > 0 \text{ if and only if } \mathbf{d}^T\mathbf{\Theta}\mathbf{d} > 0 \tag{3}$$

Note that the degree assortativity coefficient $r_{\mathbf{d}}(\mathbf{H}) = 0$.

We now turn to the analysis. To start with, we assume that consumers learn the full network $\mathbf{G}$. The monopoly's profit is proportional to the sum of Bonacich centralities, and the consumer surplus is proportional to the sum of squares of Bonacich centralities. To determine the sign of the value of network information, we thus need to compare Bonacich centralities on networks $\mathbf{G}$ and $\mathbf{H}$, both of which have the same degree distribution. In general, either positive or negative information values are possible outcomes.

We illustrate how things work on a simple example consisting of two eight-consumer networks (the example is only illustrative and does not match the theory, since the results in this paper apply to large networks). Fig. 1 depicts two networks $\mathbf{G}_A$ and $\mathbf{G}_B$ with the same degree distribution. Four consumers have two neighbors, four have three neighbors. Black (resp. white) nodes represent degree-3 (resp. degree-2) consumers. Both networks $\mathbf{G}_A$ and $\mathbf{G}_B$ have the same degree distribution. For $\delta = 0.2$, network $\mathbf{G}_A$ represented in Fig. 1-Left (resp. $\mathbf{G}_B$ in Fig. 1-Right) generates a negative (resp. positive) profit gap. In network $\mathbf{G}_A$ (resp. $\mathbf{G}_B$), consumers with the highest degrees decrease (resp. increase) their consumption level with information, while consumers with the lowest degrees increase (resp. decrease) their consumption level. In total, in network $\mathbf{G}_A$ (resp. network $\mathbf{G}_B$) information generates a decrease (resp. increase) in aggregate consumption and profit. It can be seen that in network $\mathbf{G}_A$, high-degree consumers tend to be linked with low-degree consumers, while in network $\mathbf{G}_B$ high-degree consumers tend to be linked with high-degree consumers. This suggests that degree assortativity may play a role.

To further grasp this intuition, assume that δ is close to zero. In order to assess the variation in consumption, we need to compare aggregate Bonacich centralities in networks $\mathbf{G}$ and $\mathbf{H}$. A consumer's impact on others is mainly given by his degree (for very low values of δ, the influence on the behaviors of neighbors' neighbors is much weaker than the influence on neighbors' behaviors). Since networks $\mathbf{H}$ and $\mathbf{G}$ have the same sum of degrees, aggregate demands are identical under

[10] The coefficient is defined by Newman (2002), see Appendix E for details.

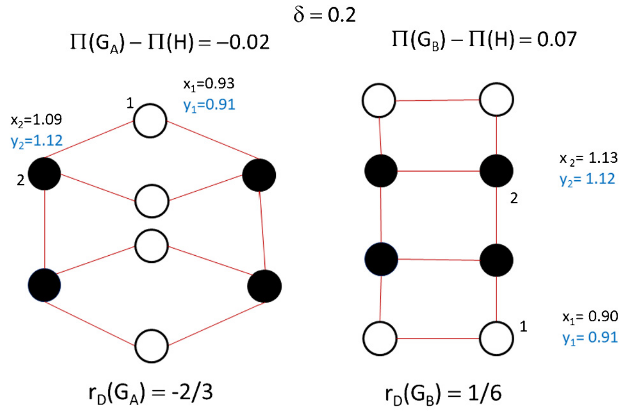


Fig. 1. Two networks with the same number of consumers and the same degree distributions; black consumers have three neighbors, white consumers two; $\delta = 0.2$; y (resp. x) codes for consumption under the initial (resp. complete) information set. For network $\mathbf{G}_A$, both degree assortativity coefficient and profit variation are negative while for network $\mathbf{G}_B$ both are positive.

Taylor approximation of order 1. At order two, a consumer's impact on others is given by the sum of her neighbors' degrees. Again, for the same degree distribution, the sum of neighbors' degrees over all agents is identical in both networks **H** and **G**, which means that both demands are also identical at order two. Hence we need to consider order three. The sum, over all agents, of the degrees of neighbors' neighbors is equal to the number of walks of length three, which is related to the coefficient of degree assortativity. Indeed, by Lemma 1, $r_{\mathbf{D}}(\mathbf{G})$ is positive whenever the difference between the number of walks of length three in network **G** and network **H** is positive. This means that, when network **G** is degree assortative, the aggregate demand increases. The same reasoning applies for consumer surplus. When the previous analysis is applied to network **V** instead of **G**, we obtain:

Proposition 2. *Under sufficiently low intensity of interaction δ, both profit and consumer surplus increase with information arrival if and only if network* **V** *is degree assortative.*

By Proposition 2, the sign of the degree assortativity coefficient coincides with the sign of outcome variations under sufficiently low intensities of interaction. For higher intensities of interaction, however, walks of lengths greater than 3 can no longer be ignored, but degree assortativity is still relevant. We now consider general intensities of interaction. The next theorem shows that, when the matrix of interaction **V** is degree assortative, the arrival of information increases profit and welfare:

Theorem 1. *Assume that Assumptions 1 and 2 hold. Denote by* **V** *the matrix of interaction of the game with enriched information on the network structure. When $\mathbf{\Theta d} \neq \mathbf{0}$, the profit and the consumer surplus are larger in the enriched information game than in the initial information game for all intensities of interaction if and only if network* **V** *is degree assortative (i.e., $r_{\mathbf{d}}(\mathbf{V}) \geq 0$). When $\mathbf{\Theta d} = \mathbf{0}$, information affects neither profit nor consumer surplus for all intensities of interaction.*

Theorem 1 is powerful because the assortativity condition is independent of the intensity of interaction. Moreover, it is worth mentioning that it is not possible to obtain a finer condition that is independent of the intensity of interaction, because a sufficient condition becomes necessary too when the intensity of interaction is sufficiently low.

For the class of networks satisfying $\mathbf{\Theta d} = \mathbf{0}$, the two games have the same outcomes, i.e., this class corresponds to the degenerate case in which network information does not affect decisions. A network satisfies $\mathbf{\Theta d} = \mathbf{0}$ if the average neighbors' degree is the same for all consumers. Moreover, in such networks there is no assortative mixing by degree. This class of networks includes regular networks (i.e., networks where all agents have the same degree), as well as other network structures with heterogeneous degree distributions, as illustrated by the twelve-consumer sixteen-link network depicted in Fig. 2.

Theorem 1 can be grasped by examining the correlation between variation in consumption $\mathbf{x} - \mathbf{y}$ and initial consumption $\mathbf{y}$, which is aligned with degree:

Proposition 3. *When network* **V** *is degree assortative, the correlation between variation in consumption and initial consumption is positive. Moreover, consumption variance increases with network information.*

Under degree assortativity, high-degree consumers increase their consumption on average, i.e. there is a positive correlation between variation in consumption and initial consumption. High-degree consumers being more influential than low-degree consumers in the network, this explains the increase in aggregate consumption (and thus profit), and drives the increase in average squared consumption (and thus consumer surplus).

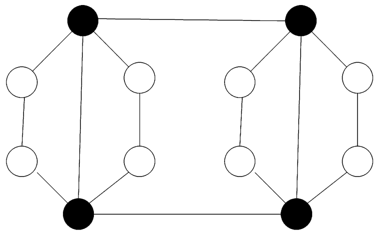


Fig. 2. A non-regular network such that $\mathbf{\Theta d} = \mathbf{0}$; black consumers have degree 4, white have degree 2.

One implication of Theorem 1 is that, in an environment where a firm possesses information on the network whereas consumers do not, it is in the firm's interest to publicly reveal the information.[11]

Actually, increased profit entails increased consumer surplus, and the magnitude of changes can be bounded from below by thresholds proportional to degree assortativity. Defining $\nu = \frac{\delta}{2} \frac{g}{g - \delta \mathbf{d}^T \mathbf{d}}$, we get:

Proposition 4. *For all intensities of interaction δ, the variation in profit and consumer surplus are bounded below by the following thresholds:*

$$\begin{cases} \pi(\mathbf{V}) - \pi(\mathbf{H}) \geq (\mathbf{d}^T \mathbf{d} - \frac{g^2}{n})\, \nu^2 \delta g \cdot r_{\mathbf{d}}(\mathbf{V}) \\ \omega(\mathbf{V}) - \omega(\mathbf{H}) \geq \left(1 + \frac{\nu}{\delta}\right)(\mathbf{d}^T \mathbf{d} - \frac{g^2}{n})\, \nu^2 \delta g \cdot r_{\mathbf{d}}(\mathbf{V}) \end{cases} \tag{4}$$

Hence, the higher the degree assortativity coefficient, the larger the lower bound on outcome variations.

It is important to stress that, following the arrival of information, a fraction of consumers may lose, depending on the intensity of interaction. However, for very high intensities of interaction, we find:

Proposition 5. *Assume that intensity of interaction δ is sufficiently high (i.e., δ tends to $\frac{1}{\mu(\mathbf{V})}$). When network* **V** *is degree assortative, network information increases the consumption (and thus the utility) of every consumer.*

5. Heterogeneous preferences

We now take into account heterogeneous private preferences for the good. This adds complexity to the model, because now consumers may differ in both their position on the network and their private preference.

We let parameter a_i represent consumer i's private preference for the good, and vector $\mathbf{a} = (a_i)_{i \in \mathcal{N}}$ the corresponding profile of preferences. Incorporating heterogeneity, the utility that agent i derives from consuming quantity q_i of the good under full network information is now given by:

$$u(q_i, q_{-i}) = a_i q_i - \frac{1}{2} q_i^2 - p_i q_i + \delta \sum_{j \in \mathcal{N}} g_{ij}\, q_i q_j \tag{5}$$

Under the incomplete information setup, each consumer knows the joint distribution of degrees and preferences, as well as her own degree and own private preference. The monopoly knows every consumer's preference and degree. In this situation, consumers are naturally typed by degree - preference (d_i, a_i). As with homogeneous preferences, under the initial information set consumers believe that there is no correlation between linked types. According to this hypothesis, the same matrix of interaction **H** as that under homogeneous preferences ensures that this heterogeneous preferences setup corresponds to a game of complete information. Note also that, in the game under enriched information, agents can acquire information about preference parameters; e.g., in the benchmark where the enriched information set corresponds to full information, they perfectly learn preference parameters.

Under preference heterogeneity, for network $\mathbf{W} \in \{\mathbf{V}, \mathbf{H}\}$, prices are still independent of the network structure but driven by individual preferences, i.e. $\mathbf{p}(\mathbf{W}) = \frac{\mathbf{a}}{2}$. Furthermore, we have $\mathbf{q}(\mathbf{W}) = \frac{1}{2}\mathbf{b_a}(\mathbf{W})$, $\pi(\mathbf{W}) = \frac{1}{2}\, \mathbf{a}^T \mathbf{q}(\mathbf{W})$, and $\omega(\mathbf{W}) = \frac{1}{8}\, \|\mathbf{b_a}(\mathbf{W})\|^2$. A key observation is that, unlike under homogeneous preferences, profit is no longer proportional to aggregate demand.

With heterogeneous preferences, it is not only degree assortativity that plays a role, but also three other assortativity coefficients (details are given in Appendix E). First, coefficient $r_{\mathbf{a}}(\mathbf{V})$ measures the level of (dis)assortative mixing by characteristic **a**. *Homophily* then refers to the case in which $r_{\mathbf{a}}(\mathbf{V}) > 0$ (i.e., $\mathbf{a}^T \mathbf{\Theta a} > 0$), and indicates that consumers are more likely

[11] This does not preclude consumers from investing in information acquisition; see for instance Leister (2019) for a related analysis of network games with similar utility functions.

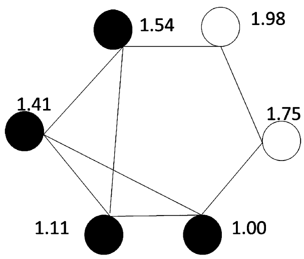


Fig. 3. The network and preferences in Example 3; numbers are the preference parameters; black nodes have degree 3, white nodes have degree 2. In this example, we have $\mathbf{V}=\mathbf{G}$, $r_{\mathbf{a}}(\mathbf{G})>0, r_{\mathbf{d}}(\mathbf{G})>0, r_{\mathbf{a},\mathbf{d}}(\mathbf{G})>-\sqrt{r_{\mathbf{a}}(\mathbf{G})r_{\mathbf{d}}(\mathbf{G})}$ but $\omega(\mathbf{G})-\omega(\mathbf{H})<0$.

to be linked to consumers with similar preferences.[12] Second, coefficient $r_{\mathbf{a},\mathbf{d}}(\mathbf{V})$ measures the tendency of high-preference consumers to be linked to high-degree consumers. When $r_{\mathbf{a},\mathbf{d}}(\mathbf{V})>0$ (i.e., $\mathbf{a}^T\mathbf{\Theta d}>0$), there is *preference-degree assortativity*. Last, coefficient $r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})$ measures the tendency of high-preference consumers to be linked to high-Bonacich centrality consumers. When $r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})>0$ (i.e., $\mathbf{a}^T\mathbf{\Theta b}(\mathbf{V})>0$), there is *preference-Bonacich centrality assortativity*.

We first address profit and then consumer surplus. A legitimate question is whether having degree assortativity and homophily simultaneously guarantees that profit increases with information. Simple examples show that it doesn't: when high-preference consumers are connected to low-degree consumers, there can be a negative information value. The next proposition shows that the coefficient of assortative mixing between preferences and degrees[13] does matter:

Proposition 6. *Assume that Assumptions 1 and 2 hold. When $\mathbf{\Theta a}\neq\mathbf{0}$ or $\mathbf{\Theta d}\neq\mathbf{0}$, network information increases the profit for all intensities of interaction δ when the following three conditions on network $\mathbf{V}$ hold simultaneously: degree assortativity ($r_{\mathbf{d}}(\mathbf{V})\geq 0$), homophily ($r_{\mathbf{a}}(\mathbf{V})\geq 0$) and the condition $r_{\mathbf{a},\mathbf{d}}(\mathbf{V})\geq-\sqrt{r_{\mathbf{a}}(\mathbf{V})\cdot r_{\mathbf{d}}(\mathbf{V})}$. When $\mathbf{\Theta a}=\mathbf{\Theta d}=\mathbf{0}$, network information does not affect profit for all intensities of interaction δ.*

The three assortative mixing conditions given in Proposition 6 guarantee assortative mixing in initial consumption, i.e. consumers with similar consumption tend to be linked with each other, which drives the profit increase. In short, these conditions guarantee that degrees and preferences reinforce each other in such a way that the average increase in the consumption level of high-degree consumers dominates the consumption decrease of low-degree consumers. Conversely, when the disassortativity between preferences and degrees is too strong, preferences and degrees are misaligned, which can produce a decrease in profit.

Next, we look at the consumer surplus, providing conditions under which network information increases consumer surplus. Remember that coefficient $r_{\mathbf{a},\mathbf{b}(\mathbf{G})}(\mathbf{V})$ measures assortative mixing between Bonacich centralities and preferences. We obtain:

Proposition 7. *Assume that Assumptions 1 and 2 hold. When $\mathbf{\Theta a}\neq\mathbf{0}$ or $\mathbf{\Theta d}\neq\mathbf{0}$, network information increases consumer surplus when the following four conditions on network $\mathbf{V}$ hold simultaneously: degree assortativity ($r_{\mathbf{d}}(\mathbf{V})\geq 0$), homophily ($r_{\mathbf{a}}(\mathbf{V})\geq 0$), the condition $r_{\mathbf{a},\mathbf{d}}(\mathbf{G})\geq-\sqrt{r_{\mathbf{a}}(\mathbf{V})\cdot r_{\mathbf{d}}(\mathbf{V})}$, and assortative mixing between Bonacich centralities and preferences ($r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})\geq 0$). When $\mathbf{\Theta a}=\mathbf{\Theta d}=\mathbf{0}$, information does not affect consumer surplus for all intensities of interaction δ.*

The key in Proposition 7 is that preference-Bonacich centrality assortativity guarantees increased demand, which in turn, combined with increased profit, guarantees increased consumer surplus. The condition $r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})\geq 0$ depends on the intensity of interaction, which means that, as parameter δ varies, the sign of coefficient $r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})$ can change.[14] It is important to stress that, when $r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V})<0$, the aggregate demand can fall, which explains the potential decrease in consumer surplus. In Example 3 below, aggregate demand decreases with network information because high-preference consumers are linked to low-degree consumers. This in turn generates a decrease in consumer surplus:

Example 3. Consider $n=6, \delta=0.1624$, $\mathbf{V}=\mathbf{G}$ and consider the network and preferences depicted in Fig. 3. We have

[12] The notion of assortative mixing by individual characteristics such as age, gender, ethnicity, etc., is also called homophily - see Lazarsfeld and Merton (1954), McPherson et al. (2001).

[13] Assortative mixing between preferences and degrees may in particular exist for types of good such that part of the preference is increasing in the ability to use them socially or to display them to others (fashion, gaming, etc.). To our knowledge, the economic literature is silent about this coefficient. Closely related, although distinct, is the correlation between valuations and number of friends (see for instance Campbell (2013) for a comparative statics exercise with respect to this coefficient).

[14] Taking $\mathbf{V}=\mathbf{G}$, we explored whether the three conditions $r_{\mathbf{a}}(\mathbf{G})\geq 0, r_{\mathbf{d}}(\mathbf{G})\geq 0, r_{\mathbf{a},\mathbf{d}}(\mathbf{G})\geq 0$ guarantee an increase in consumer surplus (for networks such that either $\mathbf{\Theta a}\neq\mathbf{0}$ or $\mathbf{\Theta d}\neq\mathbf{0}$). They do, under both sufficiently low and sufficiently high intensities of interaction. However, for intermediate intensities, the question remains open (we found no counter-example in our simulations). One difficulty is that demand can decrease under these three conditions.

$r_{\mathbf{a}}(\mathbf{G}) = 0.228$, $r_{\mathbf{d}}(\mathbf{G}) = 0.333$, $r_{\mathbf{a},\mathbf{d}}(\mathbf{G}) = -0.258 > -\sqrt{r_{\mathbf{a}}(\mathbf{G}) \cdot r_{\mathbf{d}}(\mathbf{G})} = -0.275$ and $\omega(\mathbf{G}) - \omega(\mathbf{H}) = -0.0087$. We also have $\pi(\mathbf{G}) - \pi(\mathbf{H}) = 0.1190$ and $\mathbf{1}^T(\mathbf{x} - \mathbf{y}) = -0.0119$.

This tension between profit and consumer surplus does not emerge under either very low or very high interactions. Indeed, when intensity of interaction δ is sufficiently low, a Taylor approximation at order one of the demand shows that homophily guarantees that network information increases both profit and consumer surplus. Similarly, for a sufficiently high intensity of interaction, degree assortativity guarantees that network information increases both profit and consumer surplus.[15]

For networks satisfying $\mathbf{\Theta d} = \mathbf{0}$, we have $r_{\mathbf{d}}(\mathbf{V}) = r_{\mathbf{a},\mathbf{d}}(\mathbf{V}) = r_{\mathbf{a},\mathbf{b}(\mathbf{V})}(\mathbf{V}) = 0$. Proposition 6 and Proposition 7 thus show that, when $\mathbf{\Theta a} \neq \mathbf{0}$, homophily guarantees a positive network information value (aggregate consumption is identical in the two games).

6. Discussion

We discuss alternative information structures, directed networks, and alternative pricing. For simplicity, we assume that preferences for the good are homogeneous, i.e. $\mathbf{a} = \mathbf{1}$.

Alternative information structures. So far, we have considered the arrival of public information. Our results are driven by variations in demand, i.e. learning by consumers is crucial to our analysis, otherwise the information is of no value. By contrast, the firm's learning the information is generally not crucial to establishing our results. Indeed, under linear pricing and undirected networks, prices do not depend on network structure (but do depend on preferences). However, price depends on the network structure under directed networks, or under alternative pricing (like optimal pricing, or no price discrimination). Thus, in such environments, there is a price effect as well as the demand effect. We explore this in Appendices B and C, and we confirm the key role played by assortative mixing (see the paragraphs below).

Here, we focus on the consumer information set. We perform a comparative statics on the amount of information received, by extending the model to a richer information set. We relax Assumption 1, assuming here that, initially, consumers know the degree distribution plus a public supplementary information on the network structure. Under this richer initial information set, the degree distribution of potential neighbors is not usually the same for all agents. Could profit and consumer surplus be said to grow if the arrival of information induces an increase in degree assortativity? We call the agent interaction matrix of the initial information game $\mathbf{V}^0$, and the corresponding consumption profile in the initial information game $\mathbf{y}^0$. In general, this consumption profile is Bonacich centrality, i.e. it is by no means reducible to degrees. For low intensities of interaction, previous results generalize directly. We find:

Proposition 8. *When the intensity of interaction δ is sufficiently low, and when $(\mathbf{V} - \mathbf{V}^0)\mathbf{d} \neq \mathbf{0}$, we have both $\pi(\mathbf{V}) > \pi(\mathbf{V}^0)$ and $\omega(\mathbf{V}) > \omega(\mathbf{V}^0)$ if and only if $r_{\mathbf{d}}(\mathbf{V}) \geq r_{\mathbf{d}}(\mathbf{V}^0)$.*

Proposition 8 can easily be shown by using the Taylor approximation at order three of Bonacich centralities (proof omitted). For high intensities of interaction, degree assortativity is no longer relevant. We obtain results for profit:

Proposition 9. *When $(\mathbf{V} - \mathbf{V}^0)\mathbf{y}^0 \neq \mathbf{0}$, we have $\pi(\mathbf{V}) > \pi(\mathbf{V}^0)$ for all $\delta \in]0, \frac{1}{\mu(\mathbf{V})}[$ if and only if $r_{\mathbf{y}^0}(\mathbf{V}) \geq r_{\mathbf{y}^0}(\mathbf{V}^0)$, i.e. consumers with high initial consumption levels are more likely to be linked under enriched information.*

Hence, profit increases with network information if the assortativity in initial consumption increases with the arrival of information. For consumer surplus, the same condition guarantees an increase for low and high intensities of interaction, but the result remains to be proved for intermediate intensities of interaction (we found no counter-example in our simulations).

Directed networks. We extend our analysis to directed networks (see Appendix B for more details). In the initial information set, consumers know their own in-degrees, their own out-degrees and the distributions of in-degrees and out-degrees. The main difference from the case of undirected networks is that in this more general setting, information arrival induces a price effect, because prices depend on the network. We extend Theorem 1 to directed networks as follows. We show that under degree assortativity of the - symmetric - network of averaged interaction, both profit and consumer surplus increase with information (see online Appendix B). This condition is now only sufficient in contrast to the undirected case. The main message is thus that even with directed networks, degree assortativity still guarantees increased outcomes.

Alternative pricing. We explore two alternative pricing strategies by the monopolist (see Appendix C for more details). We first examine optimal pricing where the monopolist extracts all consumer surplus. When the firm proposes prices that lead to extraction of the full consumer surplus rather than linear pricing, we find:

Proposition 11. *When the firm offers optimal prices and provided that $\mathbf{\Theta d} \neq \mathbf{0}$, network information increases profit for all intensities of interaction if and only if network $\mathbf{V}$ is degree assortative (i.e., $r_{\mathbf{d}}(\mathbf{V}) \geq 0$).*

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[15] By degree assortativity, we have that $\mu(\mathbf{V}) > \mu(\mathbf{H})$. Hence, all consumptions tend to infinity on network $\mathbf{V}$ when δ tends to $\frac{1}{\mu(\mathbf{V})}$ whereas they are still finite on network $\mathbf{H}$.

Then we explore the situation where the firm cannot price discriminate by considering a homogeneous fixed price under both the initial and the enriched information sets. This corresponds the case where the firm does not learn information about the network of consumers (i.e., only consumers learn). Clearly, the no-discrimination scenario only arises under heterogeneous preferences, since under homogeneous preferences, price is independent of position on the network. When the firm cannot price discriminate, and instead offers a homogeneous price to all consumers, we obtain that similar conditions on assortative mixing guarantee increased profit and increased consumer surplus (see online Appendix C).

7. Conclusion

In this article, we considered a monopoly selling a network good to explore whether information on the network structure is valuable to the firm and to consumers. Our analysis shows the key impact of degree assortativity, homophily, preference-degree assortativity and preference-Bonacich centrality assortativity. These results are interesting in the light of the empirically documented properties of social networks: degree assortativity and homophily.

One important insight from the above analysis is that, under heterogeneity in consumer preferences, profit and consumer surplus may vary in opposite directions following the arrival of new information. This raises the issue of the regulation of provision of information about the network structure. Our work suggests that the intensity of interaction is decisive. When the intensity of interaction is low or high, there is no need for public intervention under assortative mixing, because the assortativity conditions that guarantee increased profit also ensure increased consumer surplus. For instance, the firm will want to share some information on the network with individuals during its product launch, which is good for consumers. However, under intermediate intensities of interaction, there are circumstances under which profit increases but consumer surplus decreases.

Several questions remain open. First, it could be valuable to explore how competition between firms affects the value of information in the presence of network effects. Second, there is an issue of network evolution is an issue: for example, firms often attempt to influence the creation of opinion leaders, or to foster social relations. Lastly, consumers' investment in data protection would be an interesting line of research to pursue.

Appendix A. Proofs

Outcomes in the game under initial information. We define the matrix $\mathbf{H} = \frac{1}{g}\mathbf{d}\mathbf{d}^T$, which represents the matrix of interaction between agents under initial information. It is easily seen that the vector of consumption on network $\mathbf{H}$ is written:

$$\mathbf{y} = \frac{1}{2}\mathbf{1} + \nu\, \mathbf{d} \tag{6}$$

with

$$\nu = \frac{\delta}{2}\frac{g}{g - \delta \mathbf{d}^T\mathbf{d}} \tag{7}$$

Recalling that $g = \mathbf{d}^T\mathbf{1}$, profit and consumer surplus are given by:

$$\pi(\mathbf{H}) = \frac{n}{4} + \frac{1}{2}\nu g \tag{8}$$

and

$$\omega(\mathbf{H}) = \frac{n}{8} + \frac{1}{2}\nu g + \frac{1}{2}\nu^2\mathbf{d}^T\mathbf{d} \tag{9}$$

We present two lemmatas which, combined with Lemma 1, prove useful to establish Theorem 1, Proposition 6, and Proposition 7. Let $\mathbf{a} = (a_i)_{i\in\mathcal{N}}$ represent the profile of preferences and consumers i's utility given by equation (5). We consider an undirected network $\mathbf{V}$ and we define $\mathbf{M} = (\mathbf{I} - \delta\mathbf{V})^{-1}$. The respective consumption profiles under complete and incomplete information are $\mathbf{x} = \frac{1}{2}(\mathbf{I} - \delta\mathbf{V})^{-1}\mathbf{a}$ and $\mathbf{y} = \frac{1}{2}(\mathbf{I} - \delta\mathbf{H})^{-1}\mathbf{a}$. For every vector $\mathbf{r}$, we let $\|\mathbf{r}\|_{\mathbf{M}} = \sqrt{\mathbf{r}^T\mathbf{M}\mathbf{r}}$ denote the $\mathbf{M}$-norm of vector $\mathbf{r}$ (this is a norm as matrix $\mathbf{M}$ is positive definite), and we let $\|\mathbf{r}\|$ represent the Euclidian norm.

Lemma 2. *We have*

$$\pi(\mathbf{V}) - \pi(\mathbf{H}) = \delta^2\|\mathbf{\Theta}\mathbf{y}\|^2_{\mathbf{M}} + \delta\mathbf{y}^T\mathbf{\Theta}\mathbf{y} \tag{10}$$

Proof of Lemma 2. We have $(\mathbf{I} - \delta\mathbf{V})\mathbf{x} = (\mathbf{I} - \delta\mathbf{H})\mathbf{y}$, or equivalently, given that $\mathbf{M} = (\mathbf{I} - \delta\mathbf{V})^{-1}$,

$$\mathbf{x} - \mathbf{y} = \delta\mathbf{M}\mathbf{\Theta}\mathbf{y} \tag{11}$$

Thus, $\frac{1}{2}\mathbf{a}^T(\mathbf{x} - \mathbf{y}) = \delta\mathbf{x}^T\mathbf{\Theta}\mathbf{y}$, that is,

$$\pi(\mathbf{V}) - \pi(\mathbf{H}) = \delta \mathbf{x}^T \boldsymbol{\Theta} \mathbf{y} \tag{12}$$

Plugging $\mathbf{x} = \mathbf{y} + \delta \mathbf{M}\boldsymbol{\Theta}\mathbf{y}$ (from equation (11)) into equation (12), we obtain $\pi(\mathbf{V}) - \pi(\mathbf{H}) = \delta \mathbf{y}^T \boldsymbol{\Theta}\mathbf{y} + \delta^2 \mathbf{y}^T \boldsymbol{\Theta}^T \mathbf{M}\boldsymbol{\Theta}\mathbf{y}$. $\square$

Lemma 3. *We have*

$$\omega(\mathbf{V}) - \omega(\mathbf{H}) = \frac{\delta^2}{2} \|\mathbf{M}\boldsymbol{\Theta}\mathbf{y}\|^2 + \delta \mathbf{y}^T \mathbf{M}\boldsymbol{\Theta}\mathbf{y} \tag{13}$$

Proof of Lemma 3. Basically, $\omega(\mathbf{V}) - \omega(\mathbf{H}) = \frac{1}{2}(\mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y})$. Recalling that $\mathbf{x} = \mathbf{y} + \delta \mathbf{M}\boldsymbol{\Theta}\mathbf{y}$, we deduce that

$$\mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y} = \delta^2 (\mathbf{M}\boldsymbol{\Theta}\mathbf{y})^T (\mathbf{M}\boldsymbol{\Theta}\mathbf{y}) + 2\delta \mathbf{y}^T \mathbf{M}\boldsymbol{\Theta}\mathbf{y}$$

From equation (12), we also know that $\mathbf{1}^T(\mathbf{x} - \mathbf{y}) = \pi(\mathbf{V}) - \pi(\mathbf{H})$. The result follows directly. $\square$

Proof of Theorem 1. The proof uses Lemma 1, Lemma 2 and Lemma 3. We recall that $\mathbf{a} = \mathbf{1}$, $\mathbf{x} = \frac{1}{2}\mathbf{b}(\mathbf{V})$, and $\mathbf{y} = \frac{1}{2}\left[\mathbf{1} + \left(\frac{\delta}{1-\delta \frac{\mathbf{d}^T\mathbf{d}}{g}}\right)\mathbf{d}\right]$.

- If $\boldsymbol{\Theta}\mathbf{d} \neq \mathbf{0}$:

Monopoly profit. We prove that the condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ is sufficient. Note that $\boldsymbol{\Theta}\mathbf{d} \neq \mathbf{0}$ implies $\boldsymbol{\Theta}\mathbf{y} \neq \mathbf{0}$. By equation (10) in Lemma 2, $\pi(\mathbf{V}) - \pi(\mathbf{H}) > 0$ if

$$\mathbf{y}^T \boldsymbol{\Theta}\mathbf{y} \geq 0 \tag{14}$$

Plugging equation (6) into condition (14) and recalling that $\boldsymbol{\Theta}\mathbf{1} = \mathbf{0}$, we get $\mathbf{d}^T\boldsymbol{\Theta}\mathbf{d} \geq 0$, which is equivalent to $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ by Lemma 1.

We prove that the condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ is also necessary. Suppose that $r_{\mathbf{d}}(\mathbf{V}) < 0$. We have, for small enough δ,

$$2\mathbf{x} = \mathbf{1} + \delta\mathbf{V}\mathbf{1} + \delta^2\mathbf{V}^2\mathbf{1} + \delta^3\mathbf{V}^3\mathbf{1} + o(\delta^3)$$
$$2\mathbf{y} = \mathbf{1} + \delta\mathbf{H}\mathbf{1} + \delta^2\mathbf{H}^2\mathbf{1} + \delta^3\mathbf{H}^3\mathbf{1} + o(\delta^3)$$

We recall that $\pi(\mathbf{V}) - \pi(\mathbf{H}) = \frac{1}{2}(\mathbf{x} - \mathbf{y})$, and that, because networks $\mathbf{V}$ and $\mathbf{H}$ have the same vector of degrees, $\boldsymbol{\Theta}\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T(\mathbf{V}^2 - \mathbf{H}^2)\mathbf{1} = \mathbf{0}$. We thus obtain $\pi(\mathbf{V}) - \pi(\mathbf{H}) = \frac{1}{4}\delta^3\ \mathbf{1}^T(\mathbf{V}^3 - \mathbf{H}^3)\mathbf{1} + o(\delta^3)$. Now, we observe that, because $\mathbf{V}\mathbf{1} = \mathbf{H}\mathbf{1}$, we have $\mathbf{1}^T(\mathbf{V}^3 - \mathbf{H}^3)\mathbf{1} = \mathbf{d}^T\boldsymbol{\Theta}\mathbf{d}$. By Lemma 1, $r_{\mathbf{d}}(\mathbf{V}) < 0$ involves $\mathbf{d}^T\boldsymbol{\Theta}\mathbf{d} < 0$, and therefore $\pi(\mathbf{V}) - \pi(\mathbf{H}) < 0$ for sufficiently low δ.

Consumer surplus. We prove that the condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ is sufficient. By equation (6), $\mathbf{y} = \frac{1}{2}\mathbf{1} + \nu\mathbf{V}\mathbf{1}$. Hence, we get $\mathbf{y}^T\mathbf{M}\boldsymbol{\Theta}\mathbf{y} = \frac{1}{2}\mathbf{1}^T\mathbf{M}\boldsymbol{\Theta}\mathbf{y} + \nu\mathbf{1}^T\mathbf{G}\mathbf{M}\boldsymbol{\Theta}\mathbf{y}$. We note that $\frac{1}{2}\mathbf{1}^T\mathbf{M}\boldsymbol{\Theta}\mathbf{y} = \mathbf{x}^T\boldsymbol{\Theta}\mathbf{y}$. Moreover, because $\mathbf{x} - \frac{1}{2}\mathbf{1} = \delta\mathbf{G}\mathbf{M}\mathbf{1}$ and $\mathbf{G}\mathbf{M} = \mathbf{M}\mathbf{G}$ (since $\mathbf{V}^T = \mathbf{V}$), we have $\mathbf{1}^T\mathbf{G}\mathbf{M}\boldsymbol{\Theta}\mathbf{y} = \frac{1}{\delta}\left(\mathbf{x} - \frac{1}{2}\mathbf{1}\right)^T\boldsymbol{\Theta}\mathbf{y}$. In the end, we find that $\mathbf{y}^T\mathbf{M}\boldsymbol{\Theta}\mathbf{y} = \mathbf{x}^T\boldsymbol{\Theta}\mathbf{y} + \frac{\nu}{\delta}\left(\mathbf{x} - \frac{1}{2}\mathbf{1}\right)^T\boldsymbol{\Theta}\mathbf{y}$. Exploiting that $\boldsymbol{\Theta}^T\mathbf{1} = \boldsymbol{\Theta}\mathbf{1} = \mathbf{0}$, we obtain

$$\delta\mathbf{y}^T\mathbf{M}\boldsymbol{\Theta}\mathbf{y} = (\nu + \delta)\mathbf{x}^T\boldsymbol{\Theta}\mathbf{y} \tag{15}$$

Now, by equation (12) we have

$$\mathbf{x}^T\boldsymbol{\Theta}\mathbf{y} = \frac{\pi(\mathbf{V}) - \pi(\mathbf{H})}{\delta} \tag{16}$$

Combining equation (13) in Lemma 3, (15) and (16), we get

$$\omega(\mathbf{V}) - \omega(\mathbf{H}) = \frac{\delta^2}{2}\|\mathbf{M}\boldsymbol{\Theta}\mathbf{y}\|^2 + \left(1 + \frac{\nu}{\delta}\right)\left(\pi(\mathbf{V}) - \pi(\mathbf{H})\right) \tag{17}$$

where $\nu > 0$. Since $\boldsymbol{\Theta}\mathbf{y} \neq \mathbf{0}$, we have $\mathbf{M}\boldsymbol{\Theta}\mathbf{y} \neq \mathbf{0}$. The condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ involving a positive profit gap, we conclude by equation (17) that this also entails a positive consumer surplus gap.

We prove that the condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ is also necessary. Suppose first that $r_{\mathbf{d}}(\mathbf{V}) < 0$. Recalling that $\omega(\mathbf{V}) - \omega(\mathbf{H}) = \frac{1}{2}(\mathbf{x}^T\mathbf{x} - \mathbf{y}^T\mathbf{y})$, we obtain $\omega(\mathbf{V}) - \omega(\mathbf{H}) = \frac{1}{2}\delta^3\mathbf{1}^T(\mathbf{V}^3 - \mathbf{H}^3)\mathbf{1} + o(\delta^3)$. Note that, because $\mathbf{V}\mathbf{1} = \mathbf{H}\mathbf{1}$, we have $\mathbf{1}^T(\mathbf{V}^3 - \mathbf{H}^3)\mathbf{1} = \mathbf{d}^T\boldsymbol{\Theta}\mathbf{d}$ and thus, $r_{\mathbf{d}}(\mathbf{V}) < 0$ implies $\omega(\mathbf{V}) - \omega(\mathbf{H}) < 0$ by Lemma 6.

- If $\boldsymbol{\Theta}\mathbf{d} = \mathbf{0}$, we have that $\boldsymbol{\Theta}\mathbf{y} = \boldsymbol{\Theta}(\mathbf{1} + \nu\mathbf{d}) = \mathbf{0}$. By equation (11), we have that $\mathbf{x} = \mathbf{y}$, which implies $\pi(\mathbf{V}) = \pi(\mathbf{H})$ and $\omega(\mathbf{V}) = \omega(\mathbf{H})$. $\square$

Proof of Proposition 3. We start by showing that the correlation coefficient is positive. Let $\bar{x} = \frac{x}{n}$ and $\bar{y} = \frac{y}{n}$. Then define the coefficient $\rho = (\mathbf{x} - \mathbf{y} - (\bar{x} - \bar{y})\mathbf{1})^T(\mathbf{y} - \bar{y}\mathbf{1})$. Using equation (11), we get $\rho = \delta \mathbf{y}^T\Theta\mathbf{M}(\mathbf{y} - \bar{y}\mathbf{1})$. Exploiting that $\mathbf{y}^T\Theta\mathbf{M}\mathbf{y} = \frac{1}{2}\left(1 + \frac{1}{1-\delta\frac{\mathbf{d}^T\mathbf{d}}{g}}\right)\mathbf{x}^T\Theta\mathbf{y}$ and that $\mathbf{x} = \frac{1}{2}\mathbf{M}\mathbf{1}$, we get $\rho = \frac{1}{2}\left(\frac{1-\delta d}{1-\delta\frac{\mathbf{d}^T\mathbf{d}}{g}}\right)\mathbf{x}^T\Theta\mathbf{y}$. Now, $r_{\mathbf{d}}(\mathbf{V}) > 0$ implies $\mathbf{x}^T\Theta\mathbf{y} > 0$.

We then show that the variance is positive. The variance of vector $\mathbf{x}$ is written $Var(\mathbf{x}) = \mathbf{x}^T\mathbf{x} - \frac{(\mathbf{1}^T\mathbf{x})^2}{n}$. As $\mathbf{x} = \mathbf{y} + \delta\mathbf{M}\Theta\mathbf{y}$ (and thus $\mathbf{1}^T\mathbf{x} = \mathbf{1}^T\mathbf{y} + \delta\mathbf{1}^T\mathbf{M}\Theta\mathbf{y}$), we get

$$Var(\mathbf{x}) - Var(\mathbf{y}) = \delta^2 \underbrace{\left[\mathbf{y}^T\Theta\mathbf{M}^2\Theta\mathbf{y} - \frac{1}{n}(\mathbf{1}^T\mathbf{M}\Theta\mathbf{y})^2\right]}_{=Var(\mathbf{M}\Theta\mathbf{y})} + 2\delta \underbrace{\left[\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} - \frac{1}{n}(\mathbf{1}^T\mathbf{y})\mathbf{1}^T\mathbf{M}\Theta\mathbf{y}\right]}_{=\phi}$$

That is, given that $\mathbf{x} - \mathbf{y} = \delta\mathbf{M}\Theta\mathbf{y}$

$$Var(\mathbf{x}) - Var(\mathbf{y}) = Var(\mathbf{x} - \mathbf{y}) + 2\delta\,\phi$$

But we have $\phi > 0$ and $r_{\mathbf{d}}(\mathbf{V}) \geq 0$: indeed, $\phi = \mathbf{y}^T\mathbf{M}\Theta\mathbf{y} - \frac{1}{n}(\mathbf{1}^T\mathbf{y})\mathbf{x}^T\Theta\mathbf{y}$. Since $\mathbf{y} = \frac{1}{2}\mathbf{1} + \nu\mathbf{d}$, we have $\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} = (\frac{1}{2}\mathbf{1} + \nu\mathbf{d})^T\mathbf{M}\Theta\mathbf{y}$. Hence, given $\delta\mathbf{G}\mathbf{M} = \mathbf{M} - \mathbf{I}$, we get $\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} = \frac{1}{2}\mathbf{x}^T\Theta\mathbf{y} + \nu\mathbf{1}^T\frac{\mathbf{M}-\mathbf{I}}{\delta}\Theta\mathbf{y}$. Exploiting $\Theta\mathbf{1} = \mathbf{0}$, we find $\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} = \left(\frac{1}{2} + \frac{\nu}{\delta}\right)\mathbf{x}^T\Theta\mathbf{y}$. Therefore, since $\frac{\mathbf{1}^T\mathbf{y}}{n} = \frac{1}{2} + \nu d$, we obtain that $\phi = \nu\frac{1-\delta d}{\delta}\mathbf{x}^T\Theta\mathbf{y}$. Because $\delta d < 1$ (as $d \leq \mu(\mathbf{V})$), we conclude that $\phi > 0$ whenever $\mathbf{x}^T\Theta\mathbf{y} > 0$, which is itself guaranteed by the condition $r_{\mathbf{d}}(\mathbf{V}) \geq 0$. □

Proof of Proposition 4. Using equations (10) and (17), we get $\pi(\mathbf{V}) - \pi(\mathbf{H}) \geq \delta\mathbf{y}^T\Theta\mathbf{y}$ and $\omega(\mathbf{V}) - \omega(\mathbf{H}) \geq \left(1 + \frac{\nu}{\delta}\right)\left(\pi(\mathbf{V}) - \pi(\mathbf{H})\right)$. To compute the thresholds, we substitute $\mathbf{y}$ given by equation (6) and ν by equation (7) (setting $\mathbf{a} = \mathbf{1}$), and we obtain a threshold depending on the quantity $\mathbf{d}^T\Theta\mathbf{d}$. Then we use $\mathbf{d}^T\Theta\mathbf{d} = g(\mathbf{d}^T\mathbf{d} - \frac{g^2}{n})r_{\mathbf{d}}(\mathbf{V})$ as established from the definition of degree assortativity (see the online Appendix E) and the variance of degrees. The expression of lower bounds follows directly. □

Proof of Proposition 5. By the condition $r_{\mathbf{d}}(\mathbf{V}) > 0$, Theorem 1 in the article implies that the sum of Bonacich centralities is strictly larger on network $\mathbf{V}$ than on network $\mathbf{H}$ for all intensities of interaction. Moreover, for network $\mathbf{V}$ (resp. $\mathbf{H}$), Bonacich centralities tend to infinity when δ tends to $\frac{1}{\mu(\mathbf{V})}$ (resp. $\frac{1}{\mu(\mathbf{H})}$). Thus, we have $\mu(\mathbf{V}) \geq \mu(\mathbf{H})$. This implies that network information increases each individual consumption when parameter δ tends to $\frac{1}{\mu(\mathbf{V})}$. □

Proof of Proposition 6. There are two cases:

- $\Theta\mathbf{a} \neq \mathbf{0}$ or $\Theta\mathbf{d} \neq \mathbf{0}$: in this case $\Theta\mathbf{y} = \Theta(\mathbf{a} + \nu\mathbf{d}) \neq \mathbf{0}$, inducing $\|\Theta\mathbf{y}\|_{\mathbf{M}} > 0$. Then, by Lemma 2, we have that

$$\mathbf{y}^T\Theta\mathbf{y} \geq 0 \text{ implies } \pi(\mathbf{V}) - \pi(\mathbf{H}) > 0$$

We exploit $\mathbf{y} = \frac{\mathbf{a}}{2} + \nu\mathbf{d}$ (see equation (6)). Define

$$\zeta(t) = \left(\frac{\mathbf{a}}{2} + t\mathbf{d}\right)^T\Theta\left(\frac{\mathbf{a}}{2} + t\mathbf{d}\right)$$

The sufficient condition is thus expressed as $\zeta(\nu) \geq 0$. Note that parameter ν is increasing with δ, and when δ goes from 0 to its maximal bound, ν goes from 0 to infinity.

Now, by Lemma 6, $r_{\mathbf{a}}(\mathbf{V}) \geq 0$ implies $\zeta(0) \geq 0$. Moreover, $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ implies $\zeta(+\infty) \geq 0$. For intermediate values of parameter ν, note that $r_{\mathbf{d}}(\mathbf{V}) \geq 0$ entails that $\zeta(.)$ is U-shaped. The minimum is attained at $\nu^* = -\frac{1}{2}\frac{\mathbf{a}^T\Theta\mathbf{d}}{\mathbf{d}^T\Theta\mathbf{d}}$, and $\zeta(\nu^*) \geq 0$ if $r_{\mathbf{a},\mathbf{d}}(\mathbf{V}) \geq -\sqrt{r_{\mathbf{a}}(\mathbf{V}) \cdot r_{\mathbf{d}}(\mathbf{V})}$.

- When $\Theta\mathbf{a} = \Theta\mathbf{d} = \mathbf{0}$, we have $\Theta\mathbf{y} = \mathbf{0}$. Lemma 2 then involves $\pi(\mathbf{V}) = \pi(\mathbf{H})$. □

Proof of Proposition 7. There are two cases:

- $\Theta\mathbf{a} \neq \mathbf{0}$ or $\Theta\mathbf{d} \neq \mathbf{0}$: assume that $r_{\mathbf{d}}(\mathbf{V}) \geq 0$, $r_{\mathbf{a}}(\mathbf{V}) \geq 0$, $r_{\mathbf{a},\mathbf{d}}(\mathbf{V}) \geq -\sqrt{r_{\mathbf{a}}(\mathbf{V}) \cdot r_{\mathbf{d}}(\mathbf{V})}$, and $r_{\mathbf{b}(\mathbf{V}),\mathbf{a}}(\mathbf{V}) \geq 0$.

We have $\Theta\mathbf{y} = \Theta(\mathbf{a} + \nu\mathbf{d}) \neq \mathbf{0}$, so $\|\mathbf{M}\Theta\mathbf{y}\| > 0$. Then, by Lemma 3, $\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} > 0$ guarantees that $\omega(\mathbf{V}) > \omega(\mathbf{H})$. Since $\mathbf{y}^T\mathbf{M}\Theta\mathbf{y} = \mathbf{x}^T\Theta\mathbf{y} + \nu\ \mathbf{1}^T\mathbf{G}\mathbf{M}\Theta\mathbf{y}$, we need to show that $\mathbf{x}^T\Theta\mathbf{y} > 0$ and $\mathbf{1}^T\mathbf{G}\mathbf{M}\Theta\mathbf{y} > 0$.

. $\mathbf{x}^T\Theta\mathbf{y} > 0$: recall that $\pi(\mathbf{V}) - \pi(\mathbf{H}) = \delta\mathbf{x}^T\Theta\mathbf{y}$. Therefore, under the three first conditions, $\mathbf{x}^T\Theta\mathbf{y} > 0$.

. $\mathbf{1}^T\mathbf{G}\mathbf{M}\Theta\mathbf{y} > 0$: Note that $\mathbf{G}\mathbf{M} = \frac{\mathbf{M}-\mathbf{I}}{\delta}$, so $\mathbf{1}^T\mathbf{G}\mathbf{M}\Theta\mathbf{y} > 0$ whenever $\mathbf{b}(\mathbf{V})^T\Theta\mathbf{y} > 0$ (with $\mathbf{b}(\mathbf{V}) = \mathbf{M}\mathbf{1}$).

Since $\mathbf{y} = \frac{\mathbf{a}}{2} + \nu\mathbf{d}$, the inequality $\mathbf{b}(\mathbf{V})^T\Theta\mathbf{y} > 0$ is implied by $\mathbf{b}(\mathbf{V})^T\Theta\mathbf{a} > 0$ and $\mathbf{b}(\mathbf{V})^T\Theta\mathbf{d} > 0$. Now, on the one hand, $r_{\mathbf{b}(\mathbf{V}),\mathbf{a}}(\mathbf{V}) \geq 0$ means $\mathbf{a}^T\Theta\mathbf{b}(\mathbf{V}) \geq 0$. On the other hand, $r_{\mathbf{d}}(\mathbf{V}) > 0$ implies $\mathbf{d}^T\Theta\mathbf{b}(\mathbf{V}) > 0$. This stems from Theorem 1: indeed, this theorem states that profit gap is positive when $\mathbf{d}^T\Theta\mathbf{d} > 0$, and in the case of homogeneous preferences, profit gap is proportional to $\mathbf{b}(\mathbf{V})^T\Theta(\mathbf{1} + \nu\mathbf{d})$, i.e., to $\mathbf{b}(\mathbf{V})^T\Theta\mathbf{d}$.

- When $\Theta\mathbf{a} = \Theta\mathbf{d} = \mathbf{0}$, we have $\Theta\mathbf{y} = \mathbf{0}$. Lemma 3 then involves $\omega(\mathbf{V}) = \omega(\mathbf{H})$. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2020.12.008.

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