Understanding the power of Max-SAT resolution through UP-resilience

Mohamed Sami Cherif, Djamal Habet, André Abrame

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Mohamed Sami Cherif [1] Djamal Habet [1] André Abramé [2]

[1]Aix Marseille University, LIS, CNRS UMR 7020, Marseille, France [2]Logike, France

Maximum Satisfiability (Max-SAT)

Max-SAT Problem

Input: a formula Φ in Conjunctive Normal Form (CNF)
Output: the maximum (resp. minimum) number of satisfied (resp. falsified) clauses in Φ over all possible variable assignments

Branch & Bound (BnB) for Max-SAT

Binary search algorithm which maintains and constantly updates two values :

- **Upper Bound (UB):** value of the best known solution
- **Lower Bound (LB):** estimation of the best accessible solution

Cut: if LB ≥ UB then backtrack

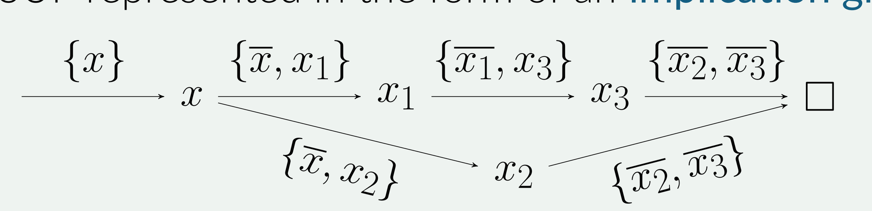
Lower Bound Estimation

At each node with current assignment I, we compute a new estimation of LB.

$$LB = FC(\Phi|_I) + IS(\Phi|_I)$$

- $FC(\Phi|_I)$: number of falsified clauses in $\Phi|_I$
- $IS(\Phi|_I)$: number of disjoint Inconsistent Subsets (IS) detected in $\Phi|_I$ by Simulated Unit Propagation (SUP)

$\psi = \{\{x\},\{\overline{x},x_1\},\{\overline{x},x_2\},\{\overline{x_1},x_3\},\{\overline{x_2},\overline{x_3}\}\}$ is an IS detected through SUP represented in the form of an implication graph :



To ensure that detected ISs are disjoint, they are temporarily deleted or transformed by Max-SAT resolution in which case they can be maintained in the current subtree.

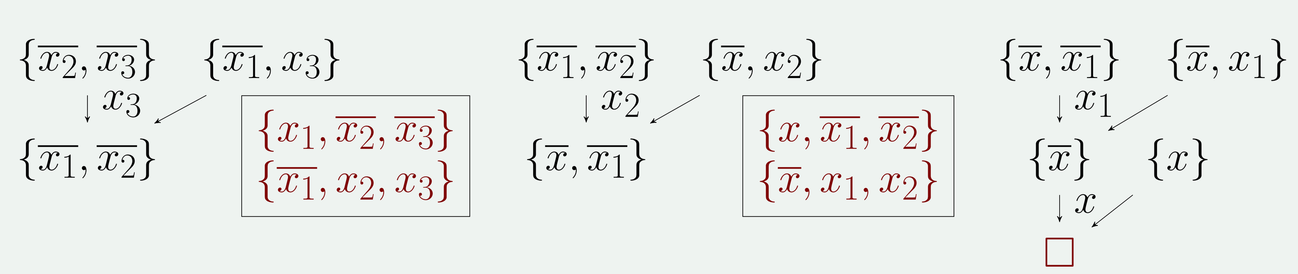
Max-SAT resolution Transformation

$$\frac{c = \{x, y_1, ..., y_s\} \quad c' = \{\overline{x}, z_1, ..., z_t\}}{cr = \{y_1, ..., y_s, z_1, ..., z_t\}, cc_1, ..., cc_t, cc_{t+1}, ..., cc_{t+s}} \text{ Max-SAT resolution}$$

Compensation clauses

$cc_1 = \{x, y_1, ..., y_s, \overline{z_1}\}$ | $cc_{t+1} = \{\overline{x}, z_1, ..., z_t, \overline{y_1}\}$
$cc_2 = \{x, y_1, ..., y_s, z_1, \overline{z_2}\}$ | $cc_{t+2} = \{\overline{x}, z_1, ..., z_t, y_1, \overline{y_2}\}$
... | ...
$cc_t = \{x, y_1, ..., y_s, z_1, ..., z_{t-1}, \overline{z_t}\}$ | $cc_{t+s} = \{\overline{x}, z_1, ..., z_t, y_1, ..., y_{s-1}, \overline{y_s}\}$

Transformation of the IS $\psi = \{\{x\},\{\overline{x},x_1\},\{\overline{x},x_2\},\{\overline{x_1},x_3\},\{\overline{x_2},\overline{x_3}\}\}$ with respect to the Reverse Propagation Order (RPO) $\langle x_3, x_2, x_1, x\rangle$:



UP-resilience

Definition

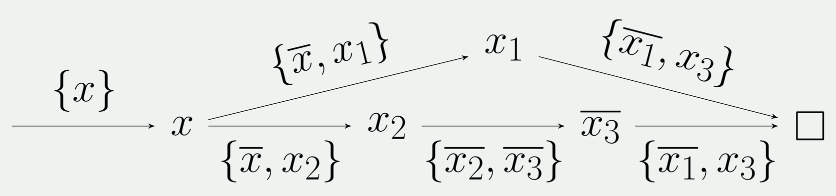
The transformation of an IS ψ is UP-resilient for a literal l in ψ iff :

$\forall \mathbf{N} \in \mathbf{pneigh}_\psi(l) : \square \in \mathbf{N}$ or l can be propagated in $\Theta(\psi)|_{\mathbf{N}}$

- $pneigh_\psi(l)$: possible neighborhood of l, i.e. the set of its different neighborhoods in all the implication graphs enabling the detection of the IS ψ
- $\Theta(\psi)|_N$: result of the Max-SAT resolution transformation of the IS ψ

The transformation of ψ is UP-resilient iff is UP-resilient for all the literals in ψ.

The IS $\psi = \{\{x\},\{\overline{x},x_1\},\{\overline{x},x_2\},\{\overline{x_1},x_3\},\{\overline{x_2},\overline{x_3}\}\}$ also corresponds to another implication graph:



- $pneigh(x_2) = \{\{x,\square\},\{x,\overline{x_3}\}\}$
- $\Theta(\psi)|_{\{x,\overline{x_3}\}} = \{\{x_1,x_2\},\{\overline{x_1},x_2\},\square\}$

→ x_2 cannot be propagated in $\Theta(\psi)|_{\{x,\overline{x_3}\}}$ (fragmentation phenomenon)
→ The transformation of ψ w.r.t RPO is not UP-resilient

Properties

- If the transformation of an IS ψ is UP-resilient for a set of literals L in ψ then $\forall N \in pneigh(L) : \square \in N$ or $\forall l \in L$, l can be propagated in $\Theta(\psi)|_{N\setminus\{l\}}$.
- The order of application (variable sequence) of Max-SAT resolution has a direct impact on the UP-resilience of the transformations.
- Efficient learning schemes (Patterns) are UP-resilient.

→ quantifies the impact of transformations on the SUP mechanism
→ provides a theoretical understanding of the efficiency of learning schemes

UP-resilience and Main Patterns

Since Max-SAT resolution transformations can negatively affect the efficiency of BnB solvers, they are only performed when they correspond to certain patterns.

$$\frac{\{l_1,l_2\},\{l_1,\overline{l_2}\}}{\{l_1\}}\,(P_1)$$

$$\frac{\{l_1,l_2\},\{l_1,l_3\},\{\overline{l_2},\overline{l_3}\}}{\{l_1\},\{l_1,l_2,l_3\},\{\overline{l_1},\overline{l_2},\overline{l_3}\}}\,(P_2)$$

$$\frac{\{l_1\},\{\overline{l_1},l_2\},\{\overline{l_2},l_3\},...,\{\overline{l_k},l_{k+1}\},\{\overline{l_{k+1}}\}}{\square,\{l_1,\overline{l_2}\},\{l_2,\overline{l_3}\},...,\{l_k,\overline{l_{k+1}}\}}\,(P_3)$$

→ Patterns do not augment the size of formula and produce unit clauses.

If a subset of an IS ψ matches the premises of pattern (P1) or (P2) or (P3). Then, the Max-SAT resolution transformation described in the pattern is UP-resilient w.r.t all possible application orders.

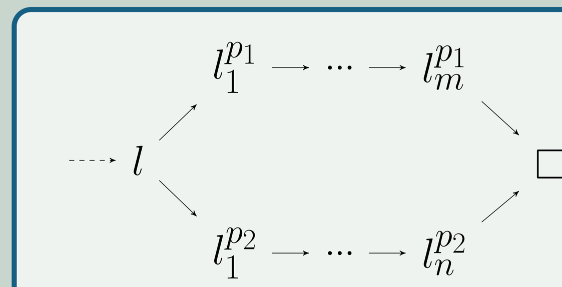
UP-resilience and Binary UCS Patterns

A binary k-Unit Clause Subset (k^b-UCS) is a set of binary clauses $\{c_1, ..., c_k\}$ in an IS such that there exists a sequence of Max-SAT resolution steps on $c_1, ..., c_k$ that produces a unit clause resolvent.

- Patterns (P_1) and (P_2) are respectively equivalent to 2^b-UCS and 3^b-UCS patterns.
- Many UCSs can be easily detected using the First Unit Implication Point (FUIP).
- RPO does not necessarily ensure the UP-resilience of k^b-UCSs for $k \geq 4$.

The subset $\psi \setminus \{x\}$ of the IS $\psi = \{\{x\},\{\overline{x},x_1\},\{\overline{x},x_2\},\{\overline{x_1},x_3\},\{\overline{x_2},\overline{x_3}\}\}$ is a 4^b-UCS, detected through the FUIP x, for which RPO does not ensure UP-resilience.

In the implication graph of ISs with k^b-UCS detected by the FUIP, there exits exactly two disjoint paths from the FUIP to $\square$.



Path Resolvent Order (PRO):
$\langle var(l_1^{p_1}), ..., var(l_m^{p_1}), var(l_1^{p_2}), ..., var(l_n^{p_2})\rangle$

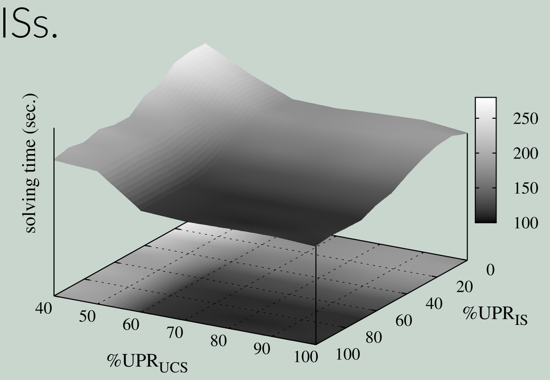
Path Resolvent Circular Order (PRCO):
$\langle var(l_1^{p_1}), ..., var(l_m^{p_1}), var(l_n^{p_2}), ..., var(l_1^{p_2})\rangle$

→ k^b-UCSs are UP-resilient with respect to PRO and PRCO
→ New approach to extend patterns by UP-resilience

Empirical Study on the Relevance of UP-resilience

- **IRS:** learns transformations whose all intermediary resolvents contain less than four literals (MiniMaxSAT learning scheme).
- **PAT:** learns the transformations when the ISs match the three main patterns.
- **PAT+:** learns the transformations when the IS matches the three main patterns or certain k-UCS patterns ($k \in \{4,5\}$).
- **UPRes:** learns only UP-resilient transformations of ISs.

Learning scheme	S (T)	D	% L	% UPR
IRS	1033 (210.69)	327864	73.9	79
PAT	1400 (72.51)	95222	21.5	100
PAT+	1402 (65.94)	72683	24.5	98
UPRes	1407 (58.27)	71279	27.6	100



1776 unweighted and weighted (partial) instances in total tested with the solver ahmaxsat, S = number of solved instances, T = average solving time, D = average number of decision, % L = percentage of learned transformations, % UPR = percentage of learned UP-resilient transformations.

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