Construction of labyrinths in pseudoconvex domains

Stéphane Charpentier, Łukasz Kosiński

CONSTRUCTION OF LABYRINTHS IN PSEUDOCONVEX DOMAINS

S. CHARPENTIER, Ł. KOSIŃSKI

Abstract. We build in a given pseudoconvex (Runge) domain D of $\mathbb{C}^N$ an $\mathcal{O}(D)$-convex set Γ, every connected component of which is a holomorphically contractible (convex) compact set, enjoying the property that any continuous path $\gamma : [0, 1) \to D$ with $\lim_{r\to 1} \gamma(r) \in \partial D$ and omitting Γ has infinite length. This solves a problem left open in a recent paper by Alarcón and Forstnerič.

1. Introduction

Alarcón and Forstnerič recently proved that the Euclidean ball $\mathbb{B}_N$ of $\mathbb{C}^N$, $N > 1$, admits a nonsingular holomorphic foliation by complete properly embedded holomorphic discs [1, Theorem 1]. They asked the natural question whether their result extends to any Runge pseudoconvex domains. As explained in [1, Remark 1], the main obstruction that appears is how to construct a *suitable labyrinth* in such a domain. Here and in the sequel we call labyrinth of a given pseudoconvex domain D in $\mathbb{C}^N$ a set Γ in D with the property that any continuous path $\gamma : [0, 1) \to D$, with $\lim_{r\to\infty} \gamma(r) \in \partial D$, whose image does not intersect Γ, has infinite length. Such sets were already built in pseudoconvex domains by Globevnik, by properly embedding the pseudoconvex domain as a submanifold of $\mathbb{C}^{2N+1}$ [7], thus reducing the problem to a construction in $\mathbb{B}_N$ [6]. However Globevnik's construction in [6, 7] did not provide with good topological properties of the connected components of the labyrinth, such as convexity or holomorphic contractibility. In [3] the authors simplified Globevnik's construction building a labyrinth in $\mathbb{B}_N$ whose connected components are balls in suitably chosen affine real hyperplanes. Alarcón and Forstnerič used a slight modification of this construction to obtain [1, Theorem 1].

The main aim of this short note is to overcome the difficulty pointed out in [1, Theorem 1] and extend the construction made in [3] to pseudoconvex domains.

Theorem 1.1. *Let D be a pseudoconvex domain in $\mathbb{C}^N$ and let (D_n) be a normal exhaustion of D by smooth strongly pseudoconvex domains that are $\mathcal{O}(D)$-convex. Let also (M_n) be a sequence of positive numbers. Then there are holomorphically contractible compact sets $\Gamma_n \subset D_{n+1} \setminus \overline{D}_n$ such that $\overline{D}_n \cup \bigcup_{j=n}^m \Gamma_j$ is $\mathcal{O}(D)$-convex for every $m \geq n$, and any continuous path connecting ∂D_n to ∂D_{n+1} and avoiding Γ_n has length greater than M_n.*

Moreover, if D is Runge then it can be additionally assumed that each connected component of Γ is the image of a $(2N-1)$-convex body under an automorphism of $\mathbb{C}^N$.

It will follow from the proof that the $(2N-1)$-convex bodies appearing in the theorem above are $\mathbb{R}$-linear transformations of $(2N-1)$-dimensional balls. Proceeding as in [1] one can use Theorem 1.1 to obtain the analogue of [1, Theorem 1] for Runge pseudoconvex domain, and thus answer the question posed by the authors. This can also be used to extend from the ball to any pseudoconvex domain some results related to Yang's problem [9, 10], such as those in [2] and [3].

2010 *Mathematics Subject Classification.* Primary 32H02 ; Secondary 32C22.

Key words and phrases. several complex variables, labyrinth.

The first author is partly supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front). The second author is partly supported by the NCN grant SONATA BIS no. 2017/26/E/ST1/00723.

The proof of Theorem 1.1 consists of three steps: firstly, to slightly adapt the main argument of [3] in order to build a suitable labyrinth in a strongly convex domain; secondly, to make use of a result by Diederich, Fornæss and Wold [4] to perform the construction in any strongly pseudoconvex domain, and thirdly, to exhaust the given pseudoconvex domain by strongly pseudoconvex ones.

2. Proof of Theorem 1.1

For a pseudoconvex domain D of $\mathbb{C}^N$ we denote by $\mathcal{O}(D)$ the space of holomorphic functions on D.

Let us first recall some classical notions and results about polynomial and holomorphic convexity. We refer the reader to Stout's book [8]. Let K and L be compact subsets of a pseudoconvex domain D of $\mathbb{C}^N$. We shall say that K and L are polynomially separated (respectively holomorphically separated with respect to D - or simply $\mathcal{O}(D)$-separated) if there exists a holomorphic polynomial p (resp. a function $f \in \mathcal{O}(D)$) such that $\widehat{p(K)} \cap \widehat{p(L)}$ is empty (resp. $\widehat{f(K)} \cap \widehat{f(L)} = \emptyset$). In particular if there exists a real hyperplane $\mathcal{H}$ such that K is contained in one of the connected component of $\mathbb{C}^N \setminus \mathcal{H}$ and L is contained in the other one, then K and L are polynomially separated. In this case, we will simply say that K and L are separated by a hyperplane. Note that if K and L are $\mathcal{O}(D)$-separated with $D = \mathbb{C}^N$, then K and L are polynomially separated.

It follows from a classical result of Kallin [8, Theorem 1.6.19], that if K and L are both polynomially convex (resp. $\mathcal{O}(D)$-convex) and polynomially separated (resp. $\mathcal{O}(D)$-separated), then $K \cup L$ is polynomially convex (resp. $\mathcal{O}(D)$-convex).

Let us now proceed with the proof of Theorem 1.1. As a first step, we shall state an analogue of [3, Theorem 1.5] for strongly convex domains.

Lemma 2.1. *Let D be a strongly convex domain in $\mathbb{C}^N$ and let $x \in \partial D$. There exists a neighborhood U of x such that for any $M > 0$ and any compact set $K \subset D$ intersecting U, there exists a compact set $\Gamma \subset D$ with the following two properties:*

(i) Γ can be written as a finite union $\bigcup_i \Gamma_i$, where each Γ_i is a convex body in a real hyperplane, and any Γ_i is separated from $\bigcup_{j<i} \Gamma_j$ by a real hyperplane.

(ii) The length of any continuous path $\gamma : [0,1) \to D \setminus \Gamma$, with $\lim_{r\to 1} \gamma(r) \in \partial U \cap D$ and such that $\gamma(r_0) \in K \cap U$ for some $r_0 \in [0,1)$ and $\gamma(r) \in U$ for any $r_0 < r < 1$, is greater than M.

We only sketch the proof, as it is a simple modification of that of [3, Theorem 1.5]. It makes use of [3, Lemma 2.1], that we recall below for notational convenience.

Lemma 2.2. *There exist numbers $m \in \mathbb{N}$, $m \geq 2$, and $c \in \mathbb{R}$, $0 < c < 1/2$, depending only on N, such that for any $r > 0$, there exist finitely many finite subsets $F_1, \dots, F_m$ of the sphere $\mathbb{S} = b\mathbb{B}_N$ which satisfy the following:*

(i) $|p - q| \geq r$ for all $p, q \in F_j$, $p \neq q$, $j = 1, \dots, m$;

(ii) If $F := \bigcup_{j=1}^m F_j$ then $F \neq \emptyset$ and $\operatorname{dist}(p, F) \leq cr$ for all $p \in \mathbb{S}$.

Outline of the proof of Lemma 2.1. Up to a translation and an $\mathbb{R}$-linear change of coordinates we can assume that $x = (1, 0, \dots, 0) \in \mathbb{C} \times \mathbb{C}^{N-1}$ and that near x a defining function r of ∂D is of the form $r(z) = \|z\|^2 - 1 + o(\|z - 1\|^2)$. Then there are open balls U_1 and U_2 centered at x and a diffeomorphism $\Phi : U_1 \to U_2$ that maps $U_1 \cap D$ onto $U_2 \cap \mathbb{B}_N$. Upon shrinking the U_i's, we can assume that Φ is arbitrarily close to the identity map in $\mathcal{C}^2$-topology. Let U be any relatively compact ball in U_1, $x \in U$.

Let m and c be given by Lemma 2.2. Following [3], we fix a sequence (s_j) of positive numbers, increasing, tending to 1 and such that $\sum_j \sqrt{s_j - s_{j-1}} = \infty$. We set $s_{j,k} := s_{j-1} +$

$k(s_j - s_{j-1})/(m+1)$. Let $S_{j,k}$ denote the set $\Phi^{-1}(U_2 \cap s_{j,k}\mathbb{S})$. Then there is a uniform constant $a > 0$ such that any tangent ball to $S_{j,k}$ of radius $r_j := a\sqrt{s_j - s_{j-1}}$ (i.e. the ball of a given radius centered at $p \in S_{j,k}$ in the real affine hyperplane tangent to $S_{j,k}$ at p) does not intersect $S_{j,k+1}$, and any ball centered at $S_{j,k} \setminus U_1$ does not intersect U. We fix $t > 1$ such that $tc < 1/2$. Let now $F_1, \ldots, F_m$ be given by Lemma 2.2 applied to $r := 2tr_j$ and denote by $E_{j,k}$ the set $\Phi^{-1}(s_{j,k}F_k \cap U_2)$. Let us denote by $\Gamma_{j,k,p}$ the tangent ball to $S_{j,k}$ centered at $p \in E_{j,k}$ and radius r_j. Observe that upon choosing Φ close enough to the identity map, in a way which depends only on $t > 1$ - hence only on N, the sets $\Gamma_{j,k,p}$ can be separated by hyperplanes from $\bigcup_{(j',k',p') \prec (j,k,p)} \Gamma_{j',k',p'}$, where $\prec$ is the lexicographical order. Following the proof from [3] one easily checks that $\Gamma := \bigcup_{j=1}^{J} \bigcup_{k,p} \Gamma_{j,k,p}$ satisfies the desired properties for some J large enough. $\square$

Remark 2.3. *(1) Observe that each connected component of Γ is the image of a $(2N-1)$-dimensional ball under an $\mathbb{R}$-affine isomorphism.*
(2) Note that setting $\Gamma := \bigcup_{j=J}^{J'} \bigcup_{k,p} \Gamma_{j,k,p}$ in the proof of Lemma 2.2, J and J' can be chosen large enough so that Γ is contained in any given ϵ-neighbourhood of ∂D and separated by a hyperplane from any given compact set in D.

The second step consists in extending [1, Theorem 1.5] to strictly pseudoconvex domains, using Lemma 2.1. This is the purpose of the next lemma.

Lemma 2.4. *Let D be a smooth strongly pseudoconvex domain in $\mathbb{C}^N$ and let K be an $\mathcal{O}(D)$-convex compact subset of D. Then for any $M > 0$ there is a compact set Γ in $D \setminus K$ such that $\Gamma \cup K$ is $\mathcal{O}(D)$-convex, with the property that any continuous path $\gamma : [0,1) \to D \setminus \Gamma$, $\gamma(0) \in K$ and $\lim_{r\to 1} \gamma(r) \in \partial D$, has length greater than M.*

Γ can be chosen so that each of its connected components is holomorphically contractible. If additionally D is Runge, the connected components of Γ can even be chosen as the images of $(2N-1)$-dimensional balls under $\mathbb{R}$-affine isomorphisms and automorphisms of $\mathbb{C}^N$.

Proof. Let K be fixed as in the statement. By [4, Theorem 1.1], for any $x \in \partial D$ there exist an open ball U centered at x and a holomorphic embedding $\Phi_x : \overline{D} \to \overline{\mathbb{B}}_N$, $\Phi_x(x) = (1, 0, \ldots, 0)$ such that $\Phi_x(U)$ and some $\Gamma' \subset \Phi_x(D)$ satisfy the conclusion of Lemma 2.1 for some constant $M' > 0$. Upon choosing M' large enough - in a way which depends only on Φ_x, the set $\Gamma := \Phi_x^{-1}(\Gamma')$ satisfies that any continuous path γ in $D \setminus \Gamma$ connecting K to $U \cap \partial D$ and satisfying $\gamma(r_0) \in K' \cap U$ for some $r_0 \in [0,1)$ and $\gamma(r) \in U$, $r_0 < r < 1$, has length greater than M. Moreover, by Remark 2.3 (2) and Kallin's theorem, Γ can also be chosen so that $\Gamma \cup K$ is $\mathcal{O}(D)$-convex and contained in an ϵ-neighbourhood of ∂D for any given $\epsilon > 0$.

Let us then consider a finite covering of ∂D by such open balls $U_{x_1}, \ldots, U_{x_k}$. Upon slightly shrinking the U_{x_j}'s, we may and shall assume that there exist $\delta, \eta > 0$ and an open η-neighbourhood V of ∂D such that the distance between $V \cap \partial U_{x_j}$ and $V \cap \partial(\bigcup_{i \neq j} U_{x_i})$ is greater than δ for $j = 1, \ldots, k$. From now on, let us fix $K' = D \setminus V$. Observe that $K' \cap U_j \neq \emptyset$ for any j and that upon choosing η small enough, we shall assume that $K \subset K'$. Let us enumerate the sets U_{x_i} as a sequence (U_j) in such a way that for any $i = 1, \ldots, k$ there exist infinitely many j such that $U_j = U_{x_i}$. We denote by Φ_j the mapping corresponding to U_j and given by [4]. Following the above procedure with $K' = D \setminus V$, we build a labyrinth Γ_1 in D and such that:

- (i) Γ_1 is contained in V and $\Gamma_1 \cup K$ is $\mathcal{O}(D)$-convex;
- (ii) Any continuous path γ in $D \setminus \Gamma_1$ connecting K to $U_1 \cap \partial D$ and satisfying $\gamma(r_0) \in K' \cap U_1$ for some $r_0 \in [0,1)$ and $\gamma(r) \in U_1$, $r_0 < r < 1$, has length greater than M.

Assuming that $\Gamma_1, \ldots, \Gamma_j$ have been built, we build Γ_{j+1} in D such that:

- (i) Γ_{j+1} is contained in an η_j-neighborhood of ∂D, where η_j is the distance from ∂D to $\Gamma_1 \cup \ldots \cup \Gamma_j$, and $\bigcup_{i=1}^{j+1} \Gamma_i \cup K$ is $\mathcal{O}(D)$-convex;

(ii) Any continuous path γ in $D \setminus \Gamma_{j+1}$ connecting K to $U_{j+1} \cap \partial D$ and satisfying $\gamma(r_0) \in K' \cap U_{j+1}$ for some $r_0 \in [0,1)$ and $\gamma(r) \in U_{j+1}$, $r_0 < r < 1$, has length greater than M.

It is now easily checked that there exists $J \in \mathbb{N}$ big enough so that $\Gamma := \bigcup_{j=1}^{J} \Gamma_j$ has the desired property. Actually, it is enough to take J such that each U_{x_j} appears in a sequence (U_j) at least n times, where $n\delta > M$. Indeed, let γ be a path in D with $\gamma(0) \in K$ and $\lim_{r\to 1} \gamma(r) \in \partial D$. Since $K \subset D \setminus V$ and γ is continuous, without loss of generality, we may assume, up to re-parametrization, that $\gamma([0,1)) \subset V$. If there exists $0 < r_1 < r_2 < 1$ and $j \leq J$ such that $\gamma(r) \in U_j$ for any $r_1 \leq r \leq r_2$ and $\gamma(r_1) \in V_{j-1}$ and $\gamma(r_2) \in V_j$, where V_j is the η_j-neighbourhood appearing in the construction, then the length of γ is clearly greater than M. If not, it means that γ has to escape from some U_j as many times as it may have to pass through some Γ_j. With J chosen as above, γ would then have to pass at least n times from a U_j to another. Since the image of γ is in V and the distance between $V \cap \partial U_{x_j}$ and $V \cap \partial(\bigcup_{i\neq j} U_{x_i})$ is greater than δ for $j = 1, \ldots, k$, the length of γ has to bigger than $n\delta$.

The $\mathcal{O}(D)$-convexity of $\Gamma \cup K$ proceeds from the construction and Kallin's theorem recalled above. Observe that the last assertion of the lemma directly follows from the construction. □

The third and last step is straightforward: Given D a pseudoconvex domain, we consider an exhaustion (D_n) of D by $\mathcal{O}(D)$-convex smooth strongly pseudoconvex domains (Runge smooth strongly pseudoconvex domains if D is Runge) and inductively apply Lemma 2.4. For the existence of such an exhaustion, we refer to [5, Subsection 2.3].

References

[1] A. Alarcón, F. Forstnerič, A foliation of the ball by complete holomorphic discs, *Math. Z.* (2019), https://doi.org/10.1007/s00209-019-02430-6.

[2] A. Alarcón, J. Globevnik, Complete embedded complex curves in the ball of $\mathbb{C}^2$ can have any topology, *Anal. PDE*, **751** (2017), 1987–1999.

[3] A. Alarcón, J. Globevnik, and F. J. Lòpez, A construction of complete complex hypersurfaces in the ball with control on the topology, *J. Reine Angew. Math.*, **751** (2019), 289–308.

[4] K. Diederich, J. E. Fornæss, E. F. Wold, Exposing Points on the Boundary of a Strictly Pseudoconvex or a Locally Convexifiable Domain of Finite 1-Type, *J. Geo. Anal.* **24** no 4 (2014), 2124–2134.

[5] F. Forstnerič, *Stein manifolds and holomorphic mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Springer, Vol. 56, 2017.

[6] J. Globevnik, A complete complex hypersurface in the ball of $\mathbb{C}^N$, *Ann. of Math.* (2) **182**:3 (2015), 1067–1091.

[7] J. Globevnik, Holomorphic functions unbounded on curves of finite length, *Math. Ann.* **364**:3-4 (2016), 1343–1359.

[8] E. Stout, *Polynomial convexity*, Progress in Mathematics, 261. Birkhäuser Boston, Inc., Boston, MA, 2007.

[9] P. Yang, Curvature of complex submanifolds in $\mathbb{C}^n$, *J. Diff. Geom.* **12** (1977), 499–511.

[10] P. Yang, Curvature of complex submanifolds in $\mathbb{C}^n$, *Proc. Symp. Pure. Math.* Vol. 30, part 2, pp. 135-137. Amer. Math. Soc., Providence, R. I. (1977).

Stéphane Charpentier, Institut de Mathematiques, UMR 7373, Aix-Marseille Universite, 39 rue F. Joliot Curie, 13453 Marseille Cedex 13, France

Email address: stephane.charpentier.1@univ-amu.fr

Łukasz Kosiński, Institute of Mathematics, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland

Email address: lukasz.kosinski@im.uj.edu.pl