Effect of Water Vorticity on Wind-Generated Gravity Waves in Finite Depth

Malek Abid, Christian Kharif

Water Waves

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--Manuscript Draft--

Manuscript Number:	WAWA-D-20-00011R2
Full Title:	Effect of water vorticity on wind-generated gravity-waves in finite depth
Article Type:	Original Article
Funding Information:	Aix-Marseille Université (AMX-19-IET-010) — Dr. Malek Abid
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Corresponding Author:	Malek Abid IRPHE FRANCE
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	IRPHE
Corresponding Author's Secondary Institution:	
First Author:	Malek Abid
First Author Secondary Information:	
Order of Authors:	Malek Abid Christian Kharif
Order of Authors Secondary Information:	
Author Comments:	Dear editor, The corrections indicated are made (in red color). Sincerely,
Response to Reviewers:	The corrections indicated are made (in red color).

Noname manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract The generation of wind waves at the surface of an established underlying vertically sheared water flow, of constant vorticity, is considered. A particular attention is paid to the role of the vorticity in water on wind-wave generation in finite depth. The present theoretical results are compared with experimental data obtained by [Young & Verhagen(1996)], in the shallow Lake George (Australia), and the least squares fit of these data by [Young (1997)]. It is shown that without vorticity in water there is a deviation between theory and experimental data. However, a good agreement between the theory and the fit of experimental data is obtained when negative vorticity is taken into account. Furthermore, it is shown that the amplitude growth rate increases with vorticity and depth. A limit to the wave energy growth, corresponding to the vanishing of the growth rate, is obtained. The corresponding limiting wave age is derived in a closed form showing its explicit dependence on vorticity and depth. The limiting wave age is found to increase with both vorticity and depth.

Keywords Shear instability · Rayleigh equation · wind-wave generation · vorticity

1 Introduction

Understanding the physics of wind-wave generation is a fundamental problem in oceanography. [Miles(1957)] is one of the first to address theoretically the transfer of wind energy to ocean surface waves. He studied the linear stability of an inviscid

M. Abid
Aix Marseille Université, CNRS, Centrale Marseille, IRPHE UMR 7342, F-13384, Marseille, France
Tel.: +33-04-13552076
Fax: +33-04-13552001
E-mail: abid@irphe.univ-mrs.fr

C. Kharif
Aix Marseille Université, CNRS, Centrale Marseille, IRPHE UMR 7342, F-13384, Marseille, France

parallel shear flow described by a boundary layer in the air above a flat surface of irrotational water of infinite depth, at rest. Miles found that the transfer of energy occurs at the critical layer level where the wind velocity equals the phase velocity of the surface waves. [Hristov, Miller & Friehe (2003)] confirmed experimentally the importance of the critical layer theory on the structure of the wave-induced air flow in open ocean. Many theoretical and numerical investigations in deep water have been devoted to this problem, among them one can cite [Valenzuela(1976)], [Kawai(1979)], [Beji & Nadaoka(2004)], [Stiassnie & *al.* (2007)] and more recently [Young & Wolfe(2014)]. For a detailed review on wind wave generation one can refer to the book by [Janssen(2004)].
Very recently, within the framework of deep water, [Kharif & Abid (2020)] found in the case of water flows of negative vorticity a limit to wave energy growth depending on the magnitude of the vorticity.
Until now there is no theoretical study on the effect of finite depth on wind-wave generation, except the paper by [Montalvo & *al.* (2013)] who considered an irrotational water flow. They considered the Miles' theory of wave amplification by wind in the case of finite depth and found that the wave growth of young waves is comparable to that of deep water whereas for old waves a finite-depth limited growth is reached.
[Young & Verhagen(1996)] conducted experiments in a shallow lake which is 20 km long by 10 km wide with an approximately uniform water depth of 2 m, and collected data set of approximately 1000 observations. From these experiments they showed that at large fetch the evolution of both the total energy and peak frequency ceases, and both parameters become depth limited. They confirmed the existence of an asymptotic limit to growth depending on the depth. However, they have not considered the possibility of the presence of vorticity in water that is likely to occur in a shallow lake due to wind at the water surface and friction at the bottom. This feature will be addressed in the sequel.

In this study we revisit the Miles theory of wind wave generation at the surface of a pre-existing underlying water flow of constant vorticity in finite depth. Constant vorticity is the first approximation that one may consider to simplify the mathematical computation to obtain preliminary results on the effect of the vorticity on wind wave generation in shallow water, and the first step toward a more general situation with depth-dependent vorticity. We consider the case of a logarithmic wind profile in the air. In section 2 the mathematical formulation is presented. Section 3 is devoted to the validation of our numerical results. The finite depth effect in the presence of vorticity is presented in section 4, as well as the comparison of our theoretical results with the experimental data of [Young & Verhagen(1996)] and the least squares fit of these data by [Young (1997)]. Our conclusions are summarized in section 5.

2 Mathematical formulation

The approach developed in what follows is similar to those of [Janssen(2004)] and [Thomas(2012)] except that we now consider a water flow of constant vorticity in finite depth. For a detailed description of the method without water vorticity in deep water one can refer to [Thomas(2012)].

The inviscid governing equations of the flow in air and water are the following

$$\nabla \cdot \mathbf{u} = \mathbf{0}, \tag{1}$$

$$\frac{d\mathbf{u}}{dt} = -\frac{\nabla p}{\rho} + \mathbf{g}, \tag{2}$$

$$\frac{d\rho}{dt} = 0, \tag{3}$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla,$$

and where $\mathbf{u}$ is the fluid velocity, ρ is the fluid density, p is the pressure and $\mathbf{g}$ is the acceleration due to gravity.
Equation (1) corresponds to mass conservation, equation (2) is the Euler equation and equation (3) means incompressible fluids.

We consider the linear stability of the following solution of the system of equations (1)-(3) which corresponds to a flat air-water interface

$$\mathbf{u} = U_0 \mathbf{e}_x, \qquad \mathbf{g} = -g\mathbf{e}_z, \qquad \rho = \rho_0(z), \qquad p_0(z) = g\int \rho_0(z)dz, \tag{4}$$

where U_0 corresponds to the velocity in the air and in the water, $\rho(z)$ corresponds to atmospheric density and water density and $\mathbf{e}_x$ and $\mathbf{e}_z$ are unit vectors in the x-direction and z-direction, respectively.

$$U_0(z) = \left\{ \begin{array}{lcr} U_a(z) & , & z > 0 \\ U_w(z) & , & -h < z < 0 \end{array} \right\}, \quad \rho_0(z) = \left\{ \begin{array}{lcr} \rho_a & , & z > 0 \\ \rho_w & , & -h < z < 0 \end{array} \right\},$$

where U_a is the wind velocity and U_w the flow velocity in the water, h is the water depth, ρ_a and ρ_w are the atmospheric density and water density, respectively. We assume the conservation of the momentum flux in the atmospheric boundary layer and the water flow is assumed to be vertically sheared with constant vorticity. Consequently The following velocity profiles will be used:

$$U_a(z) = \frac{u_*}{\kappa}\ln(1 + \frac{z}{z_0}), \qquad U_w(z) = U_s + \Omega z,$$

where u_* is the friction velocity, κ is the von Karman constant and z_0 is the roughness length of the air-water interface given by the Charnock relation $z_0 = \alpha_{ch} u_*^2/g$. For the flow velocity in the water the shear Ω and U_s are constant.

Let us perturb the equilibrium given by equations (4) with an infinitesimal perturbation

$$U = U_0 + u', \qquad p = p_0 + p', \qquad \rho = \rho_0 + \rho'. \tag{5}$$

Note that the fluid is incompressible and nonhomogenous. Therefore, the fluid density is constant on streamlines and this is ensured by equation (3). Consequently, this conservation leads to an equation for ρ'.

Substituting the expressions (5) into equations (1)-(3) and linearising gives

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0, \tag{6}$$

$$\frac{\partial u'}{\partial t}+U_0\frac{\partial u'}{\partial x}+w'U_0'(z)=-\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{7}$$

$$\frac{\partial w'}{\partial t}+U_0\frac{\partial w'}{\partial x}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial z}+\frac{\rho'}{\rho_0^2}\frac{dp_0}{dz}, \tag{8}$$

$$\frac{\partial \rho'}{\partial t}+U_0\frac{\partial \rho'}{\partial x}+w'\frac{d\rho_0}{dz}=0, \tag{9}$$

where w' is the vertical component of the velocity perturbation.
The solutions of the linearized problem are sought in the following form (normal modes)

$$[u',w',p',\rho']=[u_1(z),w_1(z),p_1(z),\rho_1(z)]\exp[i(kx-\omega t)], \tag{10}$$

where k and ω are the wavenumber and frequency of the perturbation, respectively.

Substituting the expressions (10) into the linearized equations and using $g=-p_0'/\rho_0$ gives the following Sturm-Liouville problem

$$\frac{d}{dz}\Big(\rho_0 W^2\frac{d\psi}{dz}\Big)-\Big(k^2\rho_0W^2+g\frac{d\rho_0}{dz}\Big)\psi=0, \tag{11}$$

where $W=U_0-c$, $c=\omega/k$ and $\psi=w_1/W$.

As stated above, the water flow is assumed to be vertically sheared with constant vorticity: $U_0=U_s+\Omega z$, where the shear Ω and U_s are constant. Note that the vorticity is $-\Omega$. Without loss of generality we consider a frame of reference in which $U_s=0$. Hence, equation (11) in water reads

$$\frac{d}{dz}\Big(\rho_w(\Omega z-c)^2\frac{d\psi_w}{dz}\Big)-\Big(k^2\rho_w(\Omega z-c)^2+g\frac{d\rho_w}{dz}\Big)\psi_w=0.$$

Let us assume $d\rho_w/dz=0$, then

$$(\Omega z-c)\frac{d^2\psi_w}{dz^2}+2\Omega\frac{d\psi_w}{dz}-k^2(\Omega z-c)\psi_w=0. \tag{12}$$

A critical layer in water corresponding to $\Omega z-c=0$ exists for negative values of Ω (positive vorticity). Consequently, (12) reads

$$\frac{d^2\psi_w}{dz^2}+\frac{2\Omega}{\Omega z-c}\frac{d\psi_w}{dz}-k^2\psi_w=0. \tag{13}$$

The equation (13) can be transformed to a reduced form

$$\frac{d^2\theta}{dz^2}(z)+q_1(z)\theta(z)=0, \tag{14}$$

with the following change of variables

$$\psi_w(z)=\theta(z)\exp\Big(-\frac{1}{2}\int_0^z\frac{2\Omega}{\Omega z'-c}dz'\Big),\qquad q_1(z)=-k^2-\frac{1}{2}\frac{d}{dz}\Big(\frac{2\Omega}{\Omega z-c}\Big)-\frac{1}{4}\Big(\frac{2\Omega}{\Omega z-c}\Big)^2,$$

that is

$$\psi_w = \frac{c}{c - \Omega z}\theta(z), \qquad q_1(z) = -k^2.$$

The reduced form of (13) is

$$\frac{d^2\theta}{dz^2}(z) - k^2\theta(z) = 0, \tag{15}$$

Hence, the solution of equation (13) satisfying the boundary condition $\psi_w(z = -h) = 0$ is

$$\psi_w(z) = \frac{Ac}{c - \Omega z}\frac{\sinh k(z+h)}{\cosh kh}.$$

A critical layer occurs in the water flow if the following relation is satisfied

$$c - \Omega z = 0.$$

Let $z_c = c/\Omega$ be the critical layer level. The critical layer depth in water, h_c, only occurs for $\Omega < 0$. Using the leading order of the phase velocity, c_0, given by equation (22), the critical layer depth is

$$h_c = \frac{\tanh kh}{2k} + \frac{\tanh kh}{2k}\sqrt{1 + \frac{4gk}{\Omega^2 \tanh^2 kh}}.$$

Note that this critical layer depth has been derived using the linear phase velocity, c_0, in the presence of constant vorticity and without wind. In the presence of wind, $c = c_0 + O(\rho_a/\rho_w)$. To avoid the presence of a critical layer in water we assume that $h < h_c + O(\rho_a/\rho_w)$. Due to the quite small values of Ω used, this assumption does not imply that our study is confined to extremely shallow water waves.

Equation (11) is integrated between two points below ($z = 0-$) and above ($z = 0+$) the air-water interface

$$\rho_0 W^2 \frac{d\psi}{dz}\Big|_{0-}^{0+} = \int_{0-}^{0+} \left(k^2\rho_0 W^2 + g\frac{d\rho_0}{dz}\right)\psi \, dz,$$

with

$$\frac{d\rho_0}{dz} = (\rho_w - \rho_a)\delta(z),$$

where δ is the Dirac delta function. Note that the formal relation

$$\int_{-\infty}^{+\infty} \delta(z) dz = 1,$$

indicates that δ has the physical dimension of m^{-1}.

It follows

$$\rho_a W^2(0+)\psi_a'(0+) - \rho_w W^2(0-)\psi_w'(0-) = g(\rho_a - \rho_w)\psi(0)$$

where $\psi(0) = \psi_a(0) = \psi_w(0)$ due to continuity of ψ. It results

$$\begin{aligned}
\rho_a(U_a(0+) - c)^2\psi_a'(0+) - \rho_w(\Omega z - c)^2(0-)\psi_w'(0-) &= g(\rho_a - \rho_w)\psi(0), \\
c^2(\rho_a\psi_a'(0+) - \rho_w\psi_w'(0-)) &= g(\rho_a - \rho_w)\psi(0), \\
c^2\left(\rho_a\psi_a'(0+) - \rho_w(Ak + \frac{A\Omega}{c}\tanh kh)\right) &= g(\rho_a - \rho_w)\psi(0).
\end{aligned}$$

Because we consider linear waves, without loss of generality we can set $A = 1$

$$c^2\Big(\rho_a\psi_a'(0+) - \rho_w(k + \frac{\Omega}{c}\tanh kh)\Big) = g(\rho_a - \rho_w)\tanh kh. \qquad (16)$$

If we ignore wind effect in equation (16), we obtain the linear dispersion relation of gravity waves propagating at the free surface of a flow of finite depth of constant vorticity

$$kc^2 + (\Omega\tanh kh)c - g\tanh kh = 0.$$

Let $\epsilon = \rho_a/\rho_w$ and $c = c_0 + c_1\epsilon + \mathcal{O}(\epsilon^2)$ the Taylor series in ϵ in the presence of wind (note that without wind $\epsilon = 0$). Substituting the expansion of c into equation (16) gives

At ϵ^0

$$kc_0^2 + (\Omega\tanh kh)c_0 - g\tanh kh = 0.$$

At ϵ^1

$$c_1 = \frac{c_0^2\psi_a'(0+) - g\tanh kh}{2kc_0 + \Omega\tanh kh}.$$

Following [Janssen(2004)] and [Thomas(2012)], the equation (11), in the atmospheric medium, is reduced to the following form

$$\frac{d}{dz}\Big(W_0\frac{d\psi_a}{dz}\Big) - k^2W_0^2\psi_a = 0, \qquad (17)$$
$$\psi_a(0) = 1,$$
$$\lim\psi_a(z) = 0 \quad \text{as} \quad z \to +\infty,$$

where $W_0 = U_0 - c_0$.

The growth rate γ_a of wave amplitude is

$$\gamma_a = \mathcal{I}m(kc_0 + kc_1\epsilon) = k\epsilon\mathcal{I}m(c_1),$$
$$\frac{\gamma_a}{\omega_0} = \frac{\epsilon c_0}{2kc_0 + \Omega\tanh kh}\mathcal{I}m(\psi_a'(0+)),$$

where $\omega_0 = kc_0$ and $\mathcal{I}m$ denotes imaginary part.

[Thomas(2012)] has shown that

$$\mathcal{I}m(\psi_a'(0+)) = \frac{i}{2}\mathcal{W}(\psi_a, \psi_a^*)(0+),$$

where $\mathcal{W}$ is the Wronskian given by

$$\mathcal{W}(\psi_a, \psi_a^*)(0+) = \psi_a(0+)\psi_a'^*(0+) - \psi_a'(0+)\psi_a^*(0+) = -2i\mathcal{I}m(\psi_a'(0+)),$$

and ψ_a^* denotes the complex conjugate.

Then

$$\frac{\gamma_a}{\omega_0} = i\frac{\epsilon c_0}{2(2kc_0 + \Omega\tanh kh)}\mathcal{W}(\psi_a, \psi_a^*)(0+). \qquad (18)$$

Let $\chi = w/w(0)$ be the normalised vertical component of air velocity. Then, equation (17) becomes the following Rayleigh equation

$$W_0\Big(\frac{d^2}{dz^2} - k^2\Big)\chi = W_0^{''}\chi, \qquad (19)$$

with

$$\chi(0) = 1,$$

and

$$\lim \chi(z) = 0 \quad \text{as} \quad z \to +\infty.$$

The Rayleigh equation has a singular point where the phase velocity, c_0, of the waves equals the mean wind velocity U_0. Consequently, the height, z_c, of the critical layer in the atmosphere satisfies $U_0(z_c) = c_0$.
The growth rate can be rewritten as a function of the Wronskian of the solutions of the Rayleigh equation

$$\frac{\gamma_a}{\omega_0} = i \frac{\epsilon c_0}{2(2kc_0 + \Omega \tanh kh)} \mathcal{W}(\chi, \chi^*)(0+). \tag{20}$$

One can show that $\mathcal{W}'(\chi, \chi^*) = 0$. Consequently, the Wronskian is constant for $z > z_c$ and $z < z_c$ as well and may show a jump $\mathcal{W}(z_c + \varepsilon) - \mathcal{W}(z_c - \varepsilon)$, with $\varepsilon > 0$, at the critical height. Due to the boundary condition at infinity, $\lim \mathcal{W} = 0$ as $z \to +\infty$, $\mathcal{W}(z) = 0, \forall z > z_c$. Finaly, the jump is equals to $-\mathcal{W}(z_c - \varepsilon)$ and is given by the following expression

$$-\mathcal{W}(z_c - \varepsilon) = \lim I(\Delta, \varepsilon) \quad \text{as} \quad \Delta \to 0, \Delta > 0,$$

with

$$I(\Delta, \varepsilon) = -4i \frac{W_{0c}''}{W_{0c}'} |\chi_c|^2 \arctan\left(\varepsilon \frac{W_{0c}'}{\Delta}\right).$$

The result is

$$-\mathcal{W}(z_c - \varepsilon) = -2i\pi \frac{W_{0c}''}{|W_{0c}'|} |\chi_c|^2,$$

where $W_{0c}'' = W_0''(z = z_c)$, $W_{0c}' = W_0'(z = z_c)$ and $\chi_c = \chi(z = z_c)$.
The expression of the Wronskien is

$$\mathcal{W} = 2i\pi \frac{W_{0c}''}{|W_{0c}'|} |\chi_c|^2, \quad z < z_c.$$

The normalised growth rate of surface wave amplitude is

$$\frac{\gamma_a}{\omega_0} = -\frac{\pi \epsilon c_0}{2kc_0 + \Omega \tanh kh} \frac{W_{0c}''}{|W_{0c}'|} |\chi_c|^2, \tag{21}$$

with

$$c_0 = -\frac{\Omega \tanh kh}{2k} + \sqrt{\frac{g \tanh kh}{k} + \frac{\Omega^2 \tanh^2 kh}{4k^2}}. \tag{22}$$

Equation (21) can be written differently

$$\frac{\gamma_a}{\omega_0} = -\frac{\pi \epsilon c_0}{\sqrt{4gk \tanh kh + \Omega^2 \tanh^2 kh}} \frac{W_{0c}''}{|W_{0c}'|} |\chi_c|^2. \tag{23}$$

As mentioned previously, we assume the conservation of the momentum flux in the atmospheric boundary layer. Consequently, the wind profile is given by the following logarithmic law

$$U_a(z) = \frac{u_*}{\kappa}\ln(1+\frac{z}{z_0}), \tag{24}$$

where u_* is the friction velocity, κ is the von Karman constant and z_0 is the roughness length of the air-water interface given by the Charnock relation $z_0 = \alpha_{ch}u_*^2/g$.

Within the framework of a logarithmic law we obtain

$$\frac{W_{0c}^{''}}{|W_{0c}^{'}|} = -\frac{1}{z_0}\exp(-\kappa\frac{c_0}{u_*}).$$

To derive the expression of the growth rate of the wave amplitude as a function of the wave age c_0/u_* we use as reference velocity u_* and reference length u_*^2/g. Let $c_* = c_0/u_*$, $k_* = u_*^2k/g$, $\Omega_* = u_*\Omega/g$, $h_* = gh/u_*^2$ and $z_{0*} = gz_0/u_*^2$ be the dimensionless variables and parameters. Note that $z_{0*} = \alpha_{ch}$. In dimensionless form, the growth rate of the wave amplitude is

$$\frac{\gamma_a}{\omega_0} = \frac{\rho_a}{\rho_w}\frac{\pi}{z_{0*}}\frac{c_*^2}{2-c_*\Omega_*\tanh(kh)}\exp(-\kappa c_*)|\chi_c|^2, \tag{25}$$

with

$$c_* = -\frac{\Omega_*\tanh(k_*h_*)}{2k_*} + \sqrt{\frac{\tanh(k_*h_*)}{k_*} + \frac{\Omega_*^2\tanh^2(k_*h_*)}{4k_*^2}}.$$

Note that $2 - c_*\Omega_*\tanh(k_*h_*) = 2 - \tanh(k_*h_*) + k_*c_*^2 > 1$.

Using $z_{0*} = \alpha_{ch}$, equation (25) reads

$$\frac{\gamma_a}{\omega_0} = \frac{\rho_a}{\rho_w}\frac{\pi}{\alpha_{ch}}\frac{c_*^2}{2-c_*\Omega_*\tanh(kh)}\exp(-\kappa c_*)|\chi_c|^2. \tag{26}$$

The dimensionless amplitude growth rate depends only on the wave age and vorticity.

The Rayleigh equation (19) is written in dimensionless form

$$(U_{a*} - c_*)(\frac{d^2}{dz_*^2} - k_*^2)\chi_* = U_{a*}^{''}\chi_*, \tag{27}$$

where

$$U_{a*} = \frac{1}{\kappa}\ln(1+\frac{z_*}{z_{0*}}), \qquad \chi_* = \frac{w_*}{w_*(0)}, \qquad w_* = \frac{w}{u_*}.$$

The dimensionless unknown χ_* is computed numerically by solving equation (27) with the method of [Conte & Miles(1959)]. The dimensionless growth rate of the wave amplitude γ_a/ω_0 is calculated once the critical value of χ_* is known. Rayleigh equation could be written for water. However, $U_w''(z) = 0$ cancels the singularity at $z = h_c + O(\rho_a/\rho_w)$ and consequently any instability mechanism in the water flow.

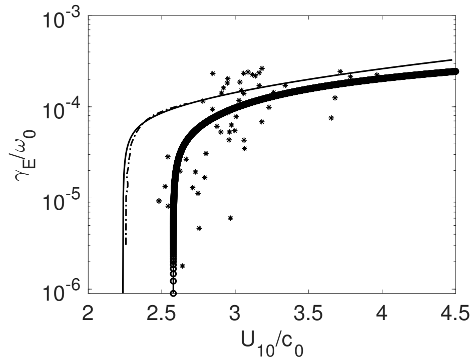

Fig. 1 Dimensionless energy growth rate as a function of the inverse of wave age without vorticity ($\Omega = 0$). Experimental data of [Young & Verhagen(1996)], $*$, and the least squares fit by [Young (1997)] $-\circ$. [Montalvo & *al.* (2013)] dot-dashed. The solid line corresponds to present results. The depth used corresponds to $\delta = gh/U_{10}^2 = 0.2$, using [Young (1997)] notation.

3 Validation

To validate our approach in finite depth we have compared our results, in the absence of vorticity ($\Omega = 0$), with numerical results of [Montalvo & *al.* (2013)], the experimental data of [Young & Verhagen(1996)], and the least squares fit by [Young (1997)]. The empirical relationship derived by [Young (1997)] to fit the experimental data reads

$$\gamma_E = A(\frac{U_{10}}{c_0} - 0.83) \tanh^n(\frac{U_{10}}{c_0} - B), \tag{28}$$

where γ_E is the energy growth rate of water waves, $A = 6.8 \times 10^{-5}$, $B = 1.25\delta^{-0.45}$ and $U_{10} \simeq 11.6u_*/\kappa$ (see [Montalvo & *al.* (2013)]). The value $n = -0.45$ was obtained by Young from a least squares fit to the data. Note that this relationship should depend on vorticity through c_0 while Young did not consider vorticity effect. Figure 1 displays the dimensionless growth rate of wave energy as a function of the inverse of wave age in the absence of vorticity. A good agreement is found between our results and those of [Montalvo & *al.* (2013)] despite the occurrence of a small deviation when the slope of the curve becomes very steep. One can observe a deviation between the result of the least squares fit of experimental data and the theoretical ones. A physical ingredient seems to be absent in the theoretical approach and will be addressed in the next section.
Note that in the limiting case of deep water ($kh \to \infty$) our results are in full agreement with those of [Beji & Nadaoka(2004)] and [Stiassnie & *al.* (2007)].

4 Vorticity effects in finite depth

Very recently, [Kharif & Abid (2020)] have shown for the infinite depth case that the amplitude growth rate, of wind generated waves, increases with vorticity except for quite old waves. They also found a limit to the wave energy growth, in the case of negative vorticity, corresponding to the vanishing of the growth rate. The

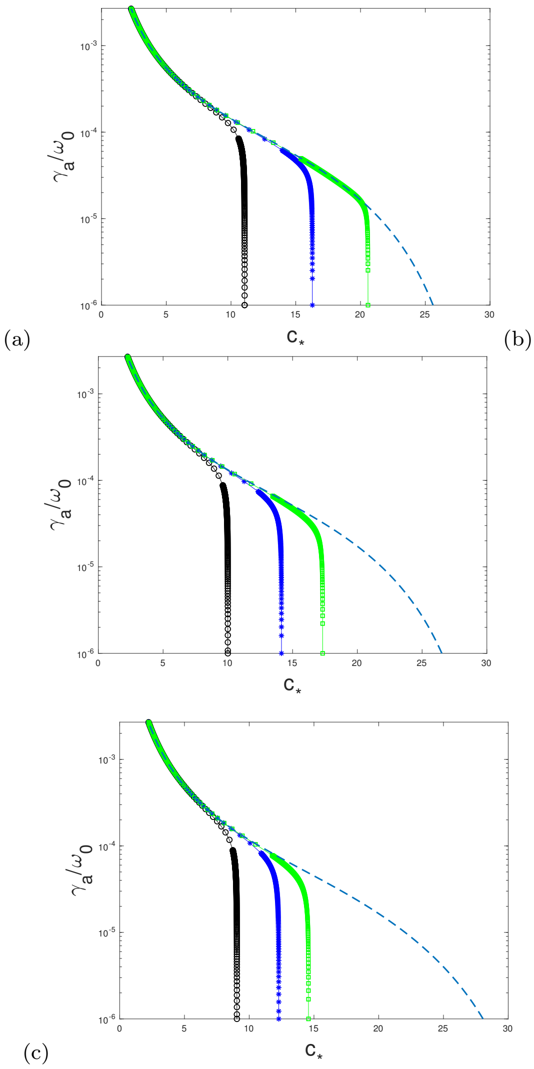

Fig. 2 (Color online) Dimensionless amplitude growth rate as a function of the wave age for different values of the dimensionless depth h_*: 100 (○), 200 ($*$), 300 (squares), ∞ (dashed) and different values of the dimensionless vorticity Ω_*: a) -0.02, b) 0, c) 0.02.

limiting wave age they found is equal to $1/\Omega_*$. We consider here the case of finite depth and find that the limit of the growth rate of waves depends not only on vorticity, Ω_*, but also on the depth h_*. Figure 2 shows the dimensionless growth rate of the wave amplitude as a function of the wave age for different values of the dimensionless depth and different values of the dimensionless vorticity. We can see that the growth rate of waves generated at the surface of a vertically sheared flow of constant vorticity decreases as the intensity Ω_* increases and vanishes when a limit to the wave amplitude growth is reached corresponding to a limiting wave

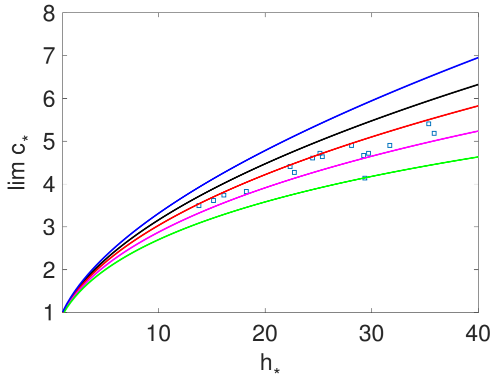

Fig. 3 (Color online) Limit phase speed as a function of depth for different values of vorticity: blue $\Omega_* = -0.03$, black $\Omega_* = 0$, red $\Omega_* = 0.0226$, magenta $\Omega_* = 0.06$ and green $\Omega_* = 0.1$. The squares correspond to AUSWEX experimental data from [Donelan & *al.* (2006)].

age. The opposite of this behavior, observed in the infinite depth case for old waves when the vorticity is positive, does not occur in finite depth. There is no wave age for which the amplitude growth rate curve with $\Omega_* < 0$ crosses the curve with $\Omega_* = 0$, in finite depth, at least for $h_* \leq 300$. Furthermore, for a given finite depth, figure 2 also shows that the limiting wave age is greater when $\Omega_* < 0$, and that the limiting wave age, for a given vorticity, increases with depth.

The wave age corresponding to $\gamma_a = 0$ can be determined easily as follows. In the limit $k \to 0$, the limiting phase speed, corresponding to the wave age with $\gamma_a = 0$, is given by (for a finite depth h)

$$\lim c_0 = -\frac{h\Omega}{2} + \sqrt{gh + \frac{h^2\Omega^2}{4}}, \qquad \text{as} \qquad k \to 0.$$

Hence,

$$\lim c_* = -\frac{h_*\Omega_*}{2} + \sqrt{h_* + \frac{h_*^2\Omega_*^2}{4}}, \qquad \text{as} \qquad k_* \to 0, \tag{29}$$

which is the dimensionless phase speed of linear long waves in the presence of constant water vorticity.
Contrarily to the case of infinite depth this expression of the limiting wave age is valid for both positive and negative vorticity. Note that without vorticity the limiting wave age is given by $\lim c_* = \sqrt{h_*}$. Hence, the growth rate of wind generated waves is depth limited in this case. When $\Omega \neq 0$, for a given finite depth, the limiting phase speed increases with positive vorticity ($\Omega < 0$) and decreases otherwise. For a given vorticity, the limiting phase speed increases with h. These features are in agreement with the results presented in figure 2 where dimensionless amplitude growth rate as a function of the wave age, for different values of the dimensionless depth and vorticity, is depicted.

The dimensionless limiting wave-age as a function of dimensionless depth, for different values of vorticity, is presented in figure 3, as well as the AUSWEX experimental data from [Donelan & *al.* (2006)]. It is clearly seen that taking the vorticity in water into account is important in determining the limiting wave age, namely in the case of negative vorticity ($\Omega_* > 0$). Indeed, the AUXEWEX data are obtained

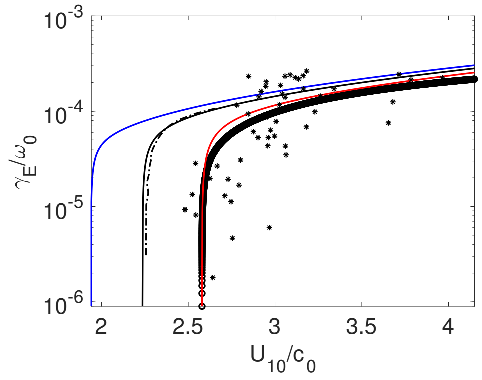

Fig. 4 (Color online) Dimensionless energy growth rate as a function of the inverse of wave age. Without vorticity ($\Omega_* = 0$): experimental data of [Young & Verhagen(1996)], $*$, and the least squares fit by [Young (1997)] $-\circ$. [Montalvo & *al.* (2013)] dot-dashed. The solid line corresponds to present results. With vorticity: red, $\Omega_* = 0.0226$, blue, $\Omega_* = -0.0226$. The depth used corresponds to $\delta = gh/U_{10}^2 = 0.2$, using [Young (1997)] notation.

from experiments on wind waves conducted in the Lake George which is a shallow lake (Australia). The combined action of wind and the friction at the bottom leads to the formation of a shear flow in the water of negative vorticity. To go further in this direction, let us revisit the experimental data of [Young & Verhagen(1996)] and the fit of [Young (1997)], presented in figure 1, using the vorticity in water. Knowing the limiting wave age and the depth we can compute the corresponding vorticity using equation (29):

$$\Omega_* = (-(\lim c_*)^2 + h_*)/(h_* \lim c_*). \tag{30}$$

From this expression, it is interesting to note that $\lim c_* = 1/\Omega_*$ when $h_* \to \infty$. From both equations (28) and (30) we have obtained the value $\Omega_* = 0.0226$. The corresponding dimensionless energy growth-rate curve, obtained with our approach, is presented in figure 4. As it can be seen in this figure the agreement between our vortical flow results and the least squares fit of the experimental data is very good, stressing the importance of taking the vorticity in water into account, namely in finite depth.

5 Conclusion

The Miles theory of wind wave generation has been revisited by considering a pre-existing underlying water flow of constant vorticity in arbitrary depth. We found that the wave energy growth rate is an increasing function of the vorticity and depth and showed that this growth rate is limited. This limit depends both on vorticity and depth. The corresponding limiting wave age is related to vorticity and depth according to the following relation

$$\Omega = \frac{-(\lim c_0)^2 + gh}{h \lim c_0},$$

and it increases with both vorticity and depth. We have compared our theoretical results with the experimental data of [Young & Verhagen(1996)] and the empirical expression derived by [Young (1997)] in the light of the dependence of the limiting wave age on vorticity. We found that water vorticity could play an important role in the growth of wind waves generated in the presence of an underlying vortical flow, whatever the depth.

To go beyond the assumption of constant vorticity, it would be interesting to extend this study to depth-dependent water vorticity and this is the object of a future work.

Acknowledgements The project leading to this publication has received funding from the Excellence Initiative of Aix-Marseille University - A*Midex, a French "Investissements d'Avenir programme" AMX-19-IET-010.

Conflict of interest

The authors declare that they have no conflict of interest.

Data availability

Data will be made available on reasonable request.

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