On homogeneous Sobolev and Besov spaces on the whole and the half space

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Abstract

In this paper, we propose an elementary construction of homogeneous Sobolev spaces of fractional order on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. This construction completes the construction of homogeneous Besov spaces on $\mathcal{S}'_h(\mathbb{R}^n)$ started by Bahouri, Chemin and Danchin on $\mathbb{R}^n$. We will also extend the treatment done by Danchin and Mucha on $\mathbb{R}^n_+$, and the construction of homogeneous Sobolev spaces of integer orders started by Danchin, Hieber, Mucha and Tolksdorf on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Properties of real and complex interpolation, duality, and density are discussed. Trace results are also reviewed. Our approach relies mostly on interpolation theory and yields simpler proofs of some already known results in the case of Besov spaces.

The lack of completeness on the whole scale will lead to consideration of intersection spaces with decoupled estimates to circumvent this issue.

As standard and simple applications, we treat the problems of Dirichlet and Neumann Laplacians in these homogeneous functions spaces.

Contents

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1 Introduction

1.1 Motivations and interests

We want to give an appropriate construction of homogeneous Sobolev spaces as subspaces of tempered distributions instead of a quotient space of distributions by polynomials. This construction is motivated by the fact that one would make sense of (para)products laws, stability under global diffeomorphism, or to look at boundary conditions, and therefore traces, when one restrict those spaces on a domain. This could be somewhat difficult if we work with tempered distributions up to a polynomial. Indeed, it is not clear that one can perform previous operations in a way that does not depend on a choice of a representative $u+P\in\mathcal{S}'(\mathbb{R}^n)$ of $[u]\in \mathcal{S}'(\mathbb{R}^n)\big/\mathbb{C}[x]$. This is inconvenient when it comes to study non-linear partial differential equations, or partial differential equations on a domain with boundary conditions. However, the interested reader could consult, for instance [BL76, Chapter 6, Section 6.3], [Tri78, Chapter 5], or [Saw18, Chapter 2, Section 2.4] for such a construction.

To circumvent those issues, the idea of Bahouri, Chemin and Danchin in [BCD11, Chapter 2] was to introduce a subspace of $\mathcal{S}'(\mathbb{R}^n)$ such that we get rid of polynomials, see [BCD11, Examples, p.23]. The aforementioned subspace of $\mathcal{S}'(\mathbb{R}^n)$ is

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u\in\mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall\Theta\in \mathrm{C}^\infty_c(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\}. \tag{1.1}$$

The condition of uniform convergence for low frequencies in above definition ensures that for $u\in\mathcal{S}'_h(\mathbb{R}^n)$, the series

$$\sum_{j\leqslant 0}\dot{\Delta}_j u$$

converge in $\mathrm{L}^\infty(\mathbb{R}^n)$, and then, [BCD11, Proposition 2.14], the following equality holds in $\mathcal{S}'(\mathbb{R}^n)$

$$u=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u,$$

where $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ is the homogeneous Littlewood-Paley decomposition on $\mathbb{R}^n$. With $\mathcal{S}'_h(\mathbb{R}^n)$ as an ambient space, Bahouri, Chemin and Danchin gave a construction of homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ which are complete whenever $(s,p,q)\in\mathbb{R}\times(1,+\infty)\times[1,+\infty]$ satisfies

$$\left[s<\frac{n}{p}\right] \text{ or } \left[q=1 \text{ and } s\leqslant\frac{n}{p}\right].$$

Later, this has also led Danchin and Mucha to consider homogeneous Besov spaces on $\mathbb{R}^n_+$ and on exterior domains, see [DM09, DM15], and Danchin, Hieber, Mucha and Tolksdorf [DHMT21] to consider homogeneous Sobolev spaces $\dot{\mathrm{H}}^{m,p}$ on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, for $m\in\mathbb{N}$, $p\in(1,+\infty)$. Each iteration led to various important applications in fluid dynamics, such as Navier-Stokes equations with variable density in [DM09, DM15], or free boundary problems as in [DHMT21]. This highlights the needs of stability under global diffeomorphism, and (para)product laws that do not rely on a choice of a representative up to a polynomial.

We want to summarize, complete and extend the given construction of Besov spaces in [BCD11, Chapter 2] and the one of homogeneous Sobolev spaces started in [DHMT21, Chapter 3]. We are going to discuss in Section 2 their construction and said usual and expected properties, and especially their behavior through complex and real interpolation. The whole space case is treated first, then the case of the half-space will follow.

Due to the lack of completeness, for homogeneous Sobolev (and Besov) spaces with high regularity exponents, one will need to consider intersection spaces $\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}$, with either, $\dot{\mathrm{H}}^{s_0,p_0}$ or $\dot{\mathrm{H}}^{s_1,p_1}$ known to be complete (*i.e.* $s_j < n/p_j$). Therefore, one will then have to check boundedness of operators with decoupled estimates.

In Section 3, we will review the meaning of traces on the boundary. As an application, in Section 4, we treat the well-posedness of Neumann and Dirichlet Laplacians on the half-space with fine enough behavior of solutions. The said "fine enough behavior" have to be understood in the sense that the decay to 0 at infinity is given a very precise sense.

1.2 Notations, definitions, usual concepts

Throughout this paper the dimension will be $n\geqslant 2$, and $\mathbb{N}$ will be the set of non-negative integers.

For two real numbers $A,B\in\mathbb{R}$, $A\lesssim_{b,c}^{a} B$ means that there exists a constant $C>0$ depending on a,b,c such that $A\leqslant CB$. When both $A\lesssim_{b,c}^{a} B$ and $B\lesssim_{b,c}^{a} A$ are true, we simply write $A\sim_{a,b,c} B$.

1.2.1 Functions spaces

Denote by $\mathcal{S}(\mathbb{R}^n,\mathbb{C})$ the space of complex valued Schwartz function, and $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ its dual called the space of tempered distributions. The Fourier transform on $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ is written $\mathcal{F}$, and is pointwise defined for any $f\in\mathrm{L}^1(\mathbb{R}^n,\mathbb{C})$ by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)\,e^{-ix\cdot\xi}\,\mathrm{d}x,\ \xi\in\mathbb{R}^n.$$

Additionnally, for $p\in[1+\infty]$, we will write $p'=\frac{p}{p-1}$ its ***Hölder conjugate***.

For any $m\in\mathbb{N}$, the map $\nabla^m : \mathcal{S}'(\mathbb{R}^n,\mathbb{C})\longrightarrow\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^{n^m})$ is defined as $\nabla^m u := (\partial^\alpha u)_{|\alpha|=m}$. We denote by $(e^{t\Delta})_{t\geqslant 0}$ and $(e^{-t(-\Delta)^{\frac{1}{2}}})_{t\geqslant 0}$ respectively the heat and Poisson semigroup on $\mathbb{R}^n$. We also introduce operators ∇' and Δ' which are respectively the gradient and the Laplacian on $\mathbb{R}^{n-1}$ identified with the $n-1$ first variables of $\mathbb{R}^n$, *i.e.* $\nabla'=(\partial_{x_1},\ldots,\partial_{x_{n-1}})$ and $\Delta'=\partial_{x_1}^2+\ldots+\partial_{x_{n-1}}^2$.

When Ω is an open set of $\mathbb{R}^n$, for $p\in[1,+\infty)$, $\mathrm{L}^p(\Omega,\mathbb{C})$ is the normed vector space of complex valued (Lebesgue-) measurable functions whose p-th power is integrable with respect to the Lebesgue measure, $\mathcal{S}(\overline{\Omega},\mathbb{C})$ (*resp.* $\mathrm{C}_c^\infty(\overline{\Omega},\mathbb{C})$) stands for functions which are restrictions on Ω of elements of $\mathcal{S}(\mathbb{R}^n,\mathbb{C})$ (*resp.* $\mathrm{C}_c^\infty(\mathbb{R}^n,\mathbb{C})$). Unless the contrary is explicitly stated, we will always identify $\mathrm{L}^p(\Omega,\mathbb{C})$ (resp. $\mathrm{C}_c^\infty(\Omega,\mathbb{C})$) as the subspace of function in $\mathrm{L}^p(\mathbb{R}^n,\mathbb{C})$ (resp. $\mathrm{C}_c^\infty(\mathbb{R}^n,\mathbb{C})$) supported in $\overline{\Omega}$ through the extension by 0 outside Ω. $\mathrm{L}^\infty(\Omega,\mathbb{C})$ stands for the space of essentially bounded (Lebesgue-) measurable functions.

For $s\in\mathbb{R}$, $p\in[1,+\infty)$, $\ell^p_s(\mathbb{Z},\mathbb{C})$, stands for the normed vector space of p-summable sequences of complexes numbers with respect to the counting measure $2^{ksp}\mathrm{d}k$; $\ell^\infty_s(\mathbb{Z},\mathbb{C})$ stands for sequences $(x_k)_{k\in\mathbb{Z}}$ such that $(2^{ks}x_k)_{k\in\mathbb{Z}}$ is bounded. More generally, when X is a Banach space, for $p\in[1,+\infty]$, one may also consider $\mathrm{L}^p(\Omega,X)$ which stands for the space of (Bochner-)measurable functions $u:\Omega\longrightarrow X$, such that $t\mapsto\|u(t)\|_X\in\mathrm{L}^p(\Omega,\mathbb{R})$, similarly one may consider $\ell^p_s(\mathbb{Z},X)$.

1.2.2 Interpolation of normed vector spaces

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two normed vector spaces. We write $X\hookrightarrow Y$ to say that X embeds continuously in Y. Now let us recall briefly basics of interpolation theory. If there exists a Hausdorff topological vector space Z, such that $X,Y\subset Z$, then $X\cap Y$ and $X+Y$ are normed vector spaces with their canonical norms, and one can define the K-functional of $z\in X+Y$, for any $t>0$ by

$$K(t,z,X,Y) := \inf_{\substack{(x,y)\in X\times Y,\\ z=x+y}} \left(\|x\|_X + t\,\|y\|_Y\right).$$

This allows us to construct, for any $\theta \in (0,1)$, $q \in [1,+\infty]$, the real interpolation spaces between X and Y with indexes θ, q as

$$(X,Y)_{\theta,q} := \left\{ x \in X+Y \,\middle|\, t \longmapsto t^{-\theta} K(t,x,X,Y) \in \mathrm{L}^q_*(\mathbb{R}_+) \right\},$$

where $\mathrm{L}^q_*(\mathbb{R}_+) := \mathrm{L}^q((0,+\infty), \mathrm{d}t/t)$. The interested reader could check [Lun18, Chapter 1], [BL76, Chapter 3] for more informations about real interpolation and its applications.

If moreover we assume that X and Y are complex Banach spaces, one can consider $\mathrm{F}(X,Y)$ the set of all continuous functions $f : \overline{S} \longmapsto X+Y$, S being the strip of complex numbers whose real part is between 0 and 1, with f holomorphic in S, and such that

$$t \longmapsto f(it) \in \mathrm{C}^0_b(\mathbb{R}, X) \quad \text{ and } \quad t \longmapsto f(1+it) \in \mathrm{C}^0_b(\mathbb{R}, Y).$$

We can endow the space $\mathrm{F}(X,Y)$ with the norm

$$\|f\|_{\mathrm{F}(X,Y)} := \max\left(\sup_{t\in\mathbb{R}} \|f(it)\|_X \, , \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y \right),$$

which makes $\mathrm{F}(X,Y)$ a Banach space since it is a closed subspace of $\mathrm{C}^0(\overline{S}, X+Y)$. Hence for $\theta \in (0,1)$, the normed vector space given by

$$[X,Y]_\theta := \left\{ f(\theta) \,\middle|\, f \in \mathrm{F}(X,Y) \right\},$$
$$\|x\|_{[X,Y]_\theta} := \inf_{\substack{f\in \mathrm{F}(X,Y),\\ f(\theta)=x}} \|f\|_{\mathrm{F}(X,Y)},$$

is a Banach space called the complex interpolation space between X and Y associated with θ. Again, the interested reader could check [Lun18, Chapter 2], [BL76, Chapter 4] for more informations about complex interpolation and its applications.

2 Homogeneous function spaces and their properties

All the function spaces considered here are scalar complex valued, hence, to alleviate notations, during this whole section we will write $\mathrm{L}^p(\Omega)$ instead of $\mathrm{L}^p(\Omega, \mathbb{C})$, and similarly for any other function spaces: we drop the arrival space $\mathbb{C}$.

2.1 Function spaces on $\mathbb{R}^n$

To deal with Besov spaces on the whole space, we need to introduce Littlewood-Paley decomposition given by $\phi \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, radial, real-valued, non-negative, such that

- supp $\phi \subset B(0, 4/3)$;
- $\phi_{|_{B(0,3/4)}} = 1$;

so we define the following functions for any $j \in \mathbb{Z}$ for all $\xi \in \mathbb{R}^n$,

$$\phi_j(\xi) := \phi(2^{-j}\xi), \qquad \psi_j(\xi) := \phi_j(\xi/2) - \phi_j(\xi),$$

and the family $(\psi_j)_{j\in\mathbb{Z}}$ has the following properties

- $\mathrm{supp}(\psi_j) \subset \{\, \xi \in \mathbb{R}^n \,|\, 3 \cdot 2^{j-2} \leqslant |\xi| \leqslant 2^{j+2}/3 \,\}$;
- $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, $\sum\limits_{j=-M}^{N} \psi_j(\xi) \xrightarrow[N,M\to+\infty]{} 1$.

Such a family $(\phi, (\psi_j)_{j\in\mathbb{Z}})$ is called a Littlewood-Paley family. Now, we consider the two following families of operators associated with their Fourier multipliers:

- The ***homogeneous*** family of Littlewood-Paley dyadic decomposition operators $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$, where

$$\dot{\Delta}_j := \mathcal{F}^{-1}\psi_j\mathcal{F},$$

- The ***inhomogeneous*** family of Littlewood-Paley dyadic decomposition operators $(\Delta_k)_{k\in\mathbb{Z}}$, where
$$\Delta_{-1} := \mathcal{F}^{-1}\phi\mathcal{F},$$
$\Delta_k := \dot{\Delta}_k$ for any $k \geqslant 0$, and $\Delta_k := 0$ for any $k \leqslant -2$.

- Lower frequency cut-off operators $(\dot{S}_j)_{j\in\mathbb{Z}}$, given for all $j \in \mathbb{Z}$ by
$$\dot{S}_j := \mathcal{F}^{-1}\phi_j\mathcal{F}.$$

One may notice, as a direct application of Young's inequality for the convolution, that they are all uniformly bounded families of operators on $\mathrm{L}^p(\mathbb{R}^n)$, $p \in [1, +\infty]$.

Both family of operators lead for $s \in \mathbb{R}$, $p, q \in [1, +\infty]$, $u \in \mathcal{S}'(\mathbb{R}^n)$ to the following quantities,
$$\|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} = \left\|(2^{ks}\,\|\Delta_k u\|_{\mathrm{L}^p(\mathbb{R}^n)})_{k\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} \text{ and } \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = \left\|(2^{js}\,\|\dot{\Delta}_j u\|_{\mathrm{L}^p(\mathbb{R}^n)})_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})},$$
respectively named the inhomogeneous and homogeneous Besov norms, but the homogeneous norm is not really a norm since $\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = 0$ does not imply that $u = 0$. Thus, following [BCD11, Chapter 2] and [DHMT21, Chapter 3], we introduce a subspace of tempered distributions such that $\|\cdot\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}$ is point-separating, say
$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall\Theta \in \mathrm{C}^\infty_c(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\},$$
where for $\lambda > 0$, $\Theta(\lambda\mathfrak{D})u = \mathcal{F}^{-1}\Theta(\lambda\cdot)\mathcal{F}u$. Notice that $\mathcal{S}'_h(\mathbb{R}^n)$ does not contain any polynomials, and for any $p \in [1, +\infty)$, $\mathrm{L}^p(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$.

One can also define the following quantities called the inhomogeneous and homogeneous Sobolev spaces' potential norms
$$\|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} := \left\|(\mathrm{I}-\Delta)^{\frac{s}{2}}u\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \text{ and } \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} := \left\|\sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u\right\|_{\mathrm{L}^p(\mathbb{R}^n)},$$
where $(-\Delta)^{\frac{s}{2}}$ is understood on $u \in \mathcal{S}'_h(\mathbb{R}^n)$ by the action on its dyadic decomposition, *i.e.*
$$(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u := \mathcal{F}^{-1}(\xi \mapsto |\xi|^s\mathcal{F}\dot{\Delta}_j u(\xi)),$$
which gives, *a priori*, a family of C^∞ functions with at most polynomial growth. Thanks to [DHMT21, Lemma 3.3, Definition 3.4],
$$\sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n)$$
holds for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$, whenever $s \in [0, +\infty)$.

When $u \in \mathcal{S}'_h(\mathbb{R}^n)$ and $\sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n)$, for $s \in \mathbb{R}$, one will simply write without distinction,
$$(-\Delta)^{\frac{s}{2}}u = \sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n),$$
which is somewhat consistent in this case with the fact that $(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u = \dot{\Delta}_j(-\Delta)^{\frac{s}{2}}u$, $j \in \mathbb{Z}$.

Hence for any $p, q \in [1, +\infty]$, $s \in \mathbb{R}$, we define

- the inhomogeneous and homogeneous Sobolev (Bessel and Riesz potential) spaces,
$\mathrm{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} < +\infty \right\}$, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \,\middle|\, \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} < +\infty \right\}$;
- and the inhomogeneous and homogeneous Besov spaces,
$\mathrm{B}^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} < +\infty \right\}$, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \,\middle|\, \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} < +\infty \right\}$,

which are all normed vector spaces. We also introduce the following closures
$$\mathcal{B}^s_{p,\infty}(\mathbb{R}^n) = \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_{\mathrm{B}^s_{p,\infty}(\mathbb{R}^n)}} \text{ and } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n) = \overline{\mathcal{S}_0(\mathbb{R}^n)}^{\|\cdot\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}}.$$

Notice that the following equalities holds with equivalence of norms for $s>0$, $p\in(1,+\infty)$, $q\in[1,+\infty]$,

$$\mathrm{L}^p(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)=\mathrm{H}^{s,p}(\mathbb{R}^n) \text{ and } \mathrm{L}^p(\mathbb{R}^n)\cap\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)=\mathrm{B}^s_{p,q}(\mathbb{R}^n),$$

see [BL76, Theorem 6.3.2] for more details.

The treatment of homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, $s\in\mathbb{R}$, $p,q\in[1,+\infty]$, defined on $\mathcal{S}'_h(\mathbb{R}^n)$ has been done in an extensive manner in [BCD11, Chapter 2]. However, the corresponding construction for homogeneous Sobolev spaces $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $s\in\mathbb{R}$, $p\in(1,+\infty)$ has only been done in the case $(p,s)\in(\{2\},\mathbb{R})\cup((1,+\infty),\mathbb{N})$. See [BCD11, Chapter 1] for the case $p=2$, [DHMT21, Chapter 3] for the case $s\in\mathbb{N}$.

We first mention the following equivalences of norms.

Proposition 2.1 *For all $s\in\mathbb{R}$, $p\in(1,+\infty)$, $q\in[1,+\infty]$, $m\in\mathbb{N}$, and all $u\in\mathcal{S}'_h(\mathbb{R}^n)$,*

$$\sum_{k=1}^{n}\|\partial^m_{x_k}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\sim_{s,m,p,n}\|\nabla^m u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\sim_{s,m,p,n}\|u\|_{\dot{\mathrm{H}}^{s+m,p}(\mathbb{R}^n)},\tag{2.1}$$

$$\sum_{j=1}^{n}\|\partial^m_{x_j}u\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)}\sim_{s,m,p,n}\|\nabla^m u\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)}\sim_{s,m,p,n}\|u\|_{\dot{\mathrm{B}}^{s+m}_{p,q}(\mathbb{R}^n)},\tag{2.2}$$

where (2.1) is a direct consequence of the proof [DHMT21, Proposition 3.7], and (2.2) a consequence of [BCD11, Lemma 2.1].

The following subspace of Schwartz functions, say

$$\mathcal{S}_0(\mathbb{R}^n):=\{\,u\in\mathcal{S}(\mathbb{R}^n)\,|\,0\notin\operatorname{supp}(\mathcal{F}f)\,\},$$

is a nice dense subspace in many cases, to be more precise

Proposition 2.2 *For all $p\in(1,+\infty)$, $q\in[1,+\infty)$, $s\in\mathbb{R}$, $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\mathrm{L}^p(\mathbb{R}^n)$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$.*

Proof. — The result for $\mathrm{L}^p(\mathbb{R}^n)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is known respectively from [DHMT21, Lemma 2.6] and [BCD11, Proposition 2.27]. The case of $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is carried over by the case $\mathrm{L}^p(\mathbb{R}^n)$. Let $s\in\mathbb{R}$, $p\in(1,+\infty)$, and $u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, then let us introduce

$$f:=(-\Delta)^{\frac{s}{2}}u\in\mathrm{L}^p(\mathbb{R}^n),$$

so from the L^p case there exists $(f_k)_{k\in\mathbb{N}}\subset\mathcal{S}_0(\mathbb{R}^n)$ such that $f_k\longrightarrow f$ in L^p. Now, for all $k\in\mathbb{N}$ we set $u_k:=(-\Delta)^{-\frac{s}{2}}f_k\in\mathcal{S}_0(\mathbb{R}^n)$, it follows

$$\|u-u_k\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}=\left\|(-\Delta)^{\frac{s}{2}}u-(-\Delta)^{\frac{s}{2}}u_k\right\|_{\mathrm{L}^p(\mathbb{R}^n)}=\|f-f_k\|_{\mathrm{L}^p(\mathbb{R}^n)}\xrightarrow[k\to+\infty]{}0.\qquad\blacksquare$$

The inhomogeneous spaces $\mathrm{L}^p(\mathbb{R}^n)$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$, and $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ are all complete for all $p,q\in[1,+\infty]$, $s\in\mathbb{R}$, but in this setting homogenenous spaces are no longer always complete (see [BCD11, Proposition 1.34, Remark 2.26]). Indeed, it can be shown (see [BCD11, Theorem 2.25]) that homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ are complete whenever $(s,p,q)\in\mathbb{R}\times(1,+\infty)\times[1,+\infty]$ satisfies

$$\left[s<\frac{n}{p}\right]\text{ or }\left[q=1\text{ and }s\leqslant\frac{n}{p}\right],\tag{$\mathcal{C}_{s,p,q}$}$$

From now, and until the end of this paper, we write $(\mathcal{C}_{s,p})$ for the statement $(\mathcal{C}_{s,p,p})$. Similarly, we show $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete whenever $(\mathcal{C}_{s,p})$ is satisfied, see Proposition 2.4 below.

To prove completeness for our homogeneous Sobolev spaces, we have to check validity of Sobolev embeddings in our setting, manually.

Proposition 2.3 *Let $p,q\in(1,+\infty)$, $s\in(0,n)$, such that*

$$\frac{1}{q}=\frac{1}{p}-\frac{s}{n}.$$

We have dense embeddings,

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}, \; \forall u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n),$$
$$\|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \; \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Proof. — Let us first recall, the fact for all $f \in \mathcal{S}(\mathbb{R}^n)$, $s \in (0,n)$, we have that $(-\Delta)^{-\frac{s}{2}} f \in \mathrm{C}^\infty(\mathbb{R}^n)$ with at most polynomial growth, in particular if $f \in \mathcal{S}_0(\mathbb{R}^n)$, we have $(-\Delta)^{-\frac{s}{2}} f \in \mathcal{S}_0(\mathbb{R}^n)$ and the Hardy-Littlewood-Sobolev inequality, see [Gra14b, Section 1.2, Theorem 1.2.3], states that for q such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, we have

$$\left\|(-\Delta)^{-\frac{s}{2}} g\right\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|g\|_{\mathrm{L}^p(\mathbb{R}^n)}, \qquad \forall g \in \mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

Therefore, by density of $\mathcal{S}_0(\mathbb{R}^n)$ in $\mathrm{L}^p(\mathbb{R}^n)$, see above Proposition 2.2, and completeness of $\mathrm{L}^q(\mathbb{R}^n)$, there exists a unique $v \in \mathrm{L}^q(\mathbb{R}^n)$, such that if $(f_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^n)$ converge to $f \in \mathrm{L}^p(\mathbb{R}^n)$, we obtain

$$(-\Delta)^{-\frac{s}{2}} f_\ell \xrightarrow[\ell\to+\infty]{\mathrm{L}^q} v,$$

then necessarily for all $k \in \mathbb{Z}$, the following convergence holds in $\mathrm{L}^q(\mathbb{R}^n)$ then in particular in $\mathcal{S}'(\mathbb{R}^n)$

$$(-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell \xrightarrow[\ell\to+\infty]{} \dot{\Delta}_k v.$$

Hence, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell, \phi \right\rangle = \left\langle \dot{\Delta}_k f_\ell, (-\Delta)^{-\frac{s}{2}} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}]\phi \right\rangle$$

so that

$$\left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell, \phi \right\rangle \xrightarrow[\ell\to+\infty]{} \left\langle \dot{\Delta}_k f, (-\Delta)^{-\frac{s}{2}} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}]\phi \right\rangle = \left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f, \phi \right\rangle.$$

Consequently, we deduce that $(-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f = \dot{\Delta}_k v$ in $\mathcal{S}'(\mathbb{R}^n)$, and since $v \in \mathrm{L}^q(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$,

$$v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j v = \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j f \in \mathrm{L}^q(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n),$$

which give the full meaning of the Hardy-Littlewood-Sobolev inequality in our setting *i.e.*,

$$\left\|(-\Delta)^{-\frac{s}{2}} u\right\|_{\mathrm{L}^q(\mathbb{R}^n)} = \|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \; \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Now if $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $(-\Delta)^{\frac{s}{2}} u \in \mathrm{L}^p(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$, and $u \in \mathcal{S}'_h(\mathbb{R}^n)$, so it follows, for all $k \in \mathbb{Z}$, that the next chain of equalities must hold pointwise,

$$\begin{aligned} \dot{\Delta}_k u &= \dot{\Delta}_k [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}] u \\ &= \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k \mathcal{F}^{-1} |\xi|^s \mathcal{F} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}] u \\ &= \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k \left[\sum_{j=k-1}^{k+1} \mathcal{F}^{-1} |\xi|^s \mathcal{F} \dot{\Delta}_j u \right], \end{aligned}$$

where we notice the property $\dot{\Delta}_k \dot{\Delta}_j = 0$, whenever $|j-k| \geqslant 2$, so that $\mathcal{F}^{-1}|\xi|^{-s}\mathcal{F}\dot{\Delta}_k \mathcal{F}^{-1}|\xi|^s \mathcal{F}\dot{\Delta}_j = 0$. Thus it comes,

$$\dot{\Delta}_k u = \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k (-\Delta)^{\frac{s}{2}} u,$$

then in $\mathcal{S}'(\mathbb{R}^n)$,

$$(-\Delta)^{-\frac{s}{2}} (-\Delta)^{\frac{s}{2}} u = \sum_{k\in\mathbb{Z}} \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k (-\Delta)^{\frac{s}{2}} u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u = u.$$

The function $(-\Delta)^{\frac{s}{2}} u$ is in $\mathrm{L}^p(\mathbb{R}^n)$ so one can apply the freshly adapted Hardy-Littlewood-Sobolev inequality to it and obtain that

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}. \qquad \blacksquare$$

Proposition 2.4 *Let $s \in \mathbb{R}$, $p \in (1, +\infty)$, then $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a Banach space whenever exponents satisfy $(\mathcal{C}_{s,p})$ (i.e. when $s < \frac{n}{p}$).*

Proof. — For $s \in \mathbb{R}$, $p \in (1, +\infty)$ satisfying $s < \frac{n}{p}$, the case $s = 0$ is already done since $\mathrm{L}^p(\mathbb{R}^n)$ is complete. Hence, we have to treat cases $s < 0$, $s \in (0, \frac{n}{p})$.

(i) **The case** $s \in (0, \frac{n}{p})$.
Now, let us consider a Cauchy sequence $(v_k)_{k\in\mathbb{N}} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, we deduce from Proposition 2.3 both that $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^q(\mathbb{R}^n)$, and $((-\Delta)^{\frac{s}{2}} v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p(\mathbb{R}^n)$. Thus, by completeness, there exists a unique couple $(v, w) \in \mathrm{L}^q(\mathbb{R}^n) \times \mathrm{L}^p(\mathbb{R}^n)$, such that

$$\|v - v_k\|_{\mathrm{L}^q(\mathbb{R}^n)} + \left\|w - (-\Delta)^{\frac{s}{2}} v_k\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0.$$

In particular, we have that $v, w \in \mathcal{S}'_h(\mathbb{R}^n)$ and by continuity, for all $j \in \mathbb{Z}$

$$(-\Delta)^{\frac{s}{2}} \dot{\Delta}_j v = \dot{\Delta}_j w$$

so that, we have the following equalities in $\mathcal{S}'(\mathbb{R}^n)$

$$(-\Delta)^{\frac{s}{2}} v = \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j w = w,$$

hence $(-\Delta)^{\frac{s}{2}} v = w \in \mathrm{L}^p(\mathbb{R}^n)$, which means exactly that $v \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, then $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete.

(ii) **The case** $s < 0$.
Let $(v_k)_{k\in\mathbb{N}} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ be a Cauchy sequence in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, by completeness of $\mathrm{L}^p(\mathbb{R}^n)$, there exists a unique $w \in \mathrm{L}^p(\mathbb{R}^n)$, such that,

$$\left\|w - (-\Delta)^{\frac{s}{2}} v_k\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0.$$

In particular, we get that $w \in \mathcal{S}'_h(\mathbb{R}^n)$. Applying [DHMT21, Lemma 3.3], we have that $(-\Delta)^{-\frac{s}{2}} w \in \mathcal{S}'_h(\mathbb{R}^n)$. Then by construction $v := (-\Delta)^{-\frac{s}{2}} w \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, and

$$\|v - v_k\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0,$$

so that $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete. ■

A direct consequence of it is the following corollary

Corollary 2.5 *Let $p \in (1, +\infty)$, $s \in \mathbb{R}$, if $(\mathcal{C}_{s,p})$ is satisfied then*

$$(-\Delta)^{\frac{s}{2}} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \longrightarrow \mathrm{L}^p(\mathbb{R}^n)$$

is an isometric isomorphism of Banach spaces.

Remark 2.6 In particular, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a reflexive Banach space, for all $p \in (1, +\infty)$, $s < n/p$.

According to [BL76, Section 6.4], for all $s \in \mathbb{R}$, $p, q \in (1, +\infty) \times [1, +\infty]$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$ and $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ are both complete, and moreover they are reflexive when $q \neq 1, +\infty$, and we have

$$(\mathrm{H}^{s,p}(\mathbb{R}^n))' = \mathrm{H}^{-s,p'}(\mathbb{R}^n),\ (\mathrm{B}^s_{p,q}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',q'}(\mathbb{R}^n), \tag{2.3}$$

$$(\mathcal{B}^s_{p,\infty}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',1}(\mathbb{R}^n),\ (\mathrm{B}^s_{p,1}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',\infty}(\mathbb{R}^n). \tag{2.4}$$

We introduce via the next lemma the equivalent homogeneous Triebel-Lizorkin norm which is somewhat important to carry over effortless usual results like the action of real and complex interpolation on our homogeneous function spaces.

Lemma 2.7 *For all $s \in \mathbb{R}$, $p \in (1, +\infty)$, let us introduce the following quantity for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,*

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} := \left\| \left\|(2^{js}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\right\|_{\ell^2(\mathbb{Z})} \right\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Then $\|\cdot\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}$ *is an equivalent norm on* $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$*, i.e. for all* $u \in \mathcal{S}'_h(\mathbb{R}^n)$*,*

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \sim_{p,n,s} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\,.$$

This is a very well known result, based on extensive use of Khintchine's inequality ($\mathrm{L}^p(\mathbb{R}^n)$ square estimates) and the Hörmander-Mikhlin Fourier multiplier theorem, but we need a proof for our specific setting, see for instance [Tri92, Remark 3, p.25] and [Gra14a, Proposition 6.1.2] for the case of $\mathcal{S}'(\mathbb{R}^n)$ when $s=0$.

Proof. — **Step 1:** $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \sim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$.

To show the inequality $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $u \in \mathrm{L}^p(\mathbb{R}^n)$ otherwise $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \leqslant +\infty$ is always true.

So if $u \in \mathrm{L}^p(\mathbb{R}^n)$, we may consider (Ω,μ) to be a probability space, and for $(\varepsilon_k)_{k\in\mathbb{Z}}$ to be a family of independent identically distributed random variables such that for all $k \in \mathbb{Z}$,

$$\mu(\{\varepsilon_k=-1\}) = \mu(\{\varepsilon_k=1\}) = \frac{1}{2}.$$

One deduce from Khintchine's inequality, see for instance either [MS13, Lemma 5.5], [KW04, Section I, Lemma 2.2] or [Gra14a, Appendix C], that

$$\begin{aligned}\|u\|^p_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} &\sim_p \int_{\mathbb{R}^n}\int_{\Omega} \Big|\sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j u(x)\Big|^p \mathrm{d}\mu(\omega)\,\mathrm{d}x\\ &\sim_p \int_{\Omega}\int_{\mathbb{R}^n} \Big|\sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j u(x)\Big|^p \mathrm{d}x\,\mathrm{d}\mu(\omega),\end{aligned}$$

where the last estimate comes from an application of Fubini-Tonelli's theorem. Hence it suffices to investigate the L^p-boundedness of the following random Fourier multiplier operator, defined for almost all $\omega \in \Omega$,

$$T(\omega) := \sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j,$$

whose Fourier symbol is given by the function $K(\omega)$, such that for all $\xi \in \mathbb{R}^n$,

$$K(\omega)(\xi) = \sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\psi(2^{-j}\xi).$$

It is not difficult to see that one can make a partition $\mathbb{R}^n$ into annulus of size $|\xi| \sim 2^j$, $j \in \mathbb{Z}$, to check that for all $\ell \in \mathbb{N}$, $\xi \in \mathbb{R}^n\setminus\{0\}$,

$$|\nabla^\ell K(\omega)(\xi)| \lesssim_{n,\ell,\psi} \frac{1}{|\xi|^\ell},$$

where the implicit constant does not depend on ω. Therefore, one may apply Hörmander-Mikhlin's Fourier multiplier Theorem to deduce that $T(\omega)$ is bounded on L^p and admits a uniform bound with respect to ω, and use the fact that $\mu(\Omega)=1$ to obtain,

$$\int_{\Omega}\int_{\mathbb{R}^n} \big|T(\omega)u(x)\big|^p \mathrm{d}x\,\mathrm{d}\mu(\omega) \lesssim_{p,n} \int_{\mathbb{R}^n} |u(x)|^p \mathrm{d}x.$$

Thus, we have obtained $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$.

Now, to prove $\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we are going to argue by duality. Let $u \in \mathcal{S}'_h(\mathbb{R}^n)$, and $v \in \mathcal{S}(\mathbb{R}^n)$. We can decompose the action of u on v as

$$\big\langle u,\, v\big\rangle_{\mathbb{R}^n} = \sum_{j\in\mathbb{Z}} \Big\langle \dot{\Delta}_j u,\, [\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}]v\Big\rangle_{\mathbb{R}^n}$$

so that by $\mathrm{L}^p(\ell^2)$-$\mathrm{L}^{p'}(\ell^2)$ Hölder's inequality, we obtain

$$\begin{aligned}\left|\langle u,\, v\rangle_{\mathbb{R}^n}\right| &\leqslant \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \left\|\|([\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}]v)_{j\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}\right\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}\\ &\leqslant 3\,\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)}\,\|v\|_{\dot{\mathrm{F}}^0_{p',2}(\mathbb{R}^n)}\,.\end{aligned}$$

One may apply the previous estimate $\|v\|_{\dot{\mathrm{F}}^0_{p',2}(\mathbb{R}^n)} \lesssim_{p',n} \|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}$, to deduce

$$\left|\langle u,\, v\rangle_{\mathbb{R}^n}\right| \lesssim_{p',n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)}\,\|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}\,.$$

Therefore, taking the supremum on $v\in\mathcal{S}(\mathbb{R}^n)$ such that $\|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}\leqslant 1$, yields

$$\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)}\,.$$

Step 2: $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \sim_{p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}$, $u\in\mathcal{S}'_h(\mathbb{R}^n)$, $s\neq 0$.

The proof starts similarly, introducing the following random Fourier multiplier operator

$$T^s(\omega) := \sum_{j\in\mathbb{Z}} 2^{js}(-\Delta)^{-\frac{s}{2}}\varepsilon_j(\omega)\dot{\Delta}_j,$$

from which one fairly obtains, for all $v\in\mathcal{S}'_h(\mathbb{R}^n)$,

$$\left\|\|(2^{js}(-\Delta)^{-\frac{s}{2}}\dot{\Delta}_j v)_{j\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \sim_{p,n,s} \|v\|_{\mathrm{L}^p(\mathbb{R}^n)}\,.$$

Now, if $u\in\mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ otherwise $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}\leqslant+\infty$ is always true. One may plug $v=(-\Delta)^{\frac{s}{2}}u=\sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u$, to obtain first, from $(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u=\dot{\Delta}_j(-\Delta)^{\frac{s}{2}}u$,

$$\left\|\|(2^{js}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n,s} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\,.$$

For the reverse estimate, similarly, provided $u\in\mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}<+\infty$ otherwise $\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\leqslant+\infty$ is always true. The Fatou lemma yields

$$\begin{aligned}\Big\|\sum_{j\in\mathbb{Z}}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u\Big\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leqslant \liminf_{N\to+\infty}\Big\|\sum_{j\in[\![-N,N]\!]}(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u\Big\|_{\mathrm{L}^p(\mathbb{R}^n)}\\ &\lesssim_{p,n,s} \liminf_{N\to+\infty}\left(\Big\|\Big(\sum_{j\in[\![-N,N]\!]}|2^{js}\dot{\Delta}_j u|^2\Big)^{\frac{1}{2}}\Big\|_{\mathrm{L}^p(\mathbb{R}^n)}\right.\\ &\qquad\left.+\,2^{-(N+1)s}\|\dot{\Delta}_{-(N+1)}u\|_{\mathrm{L}^p(\mathbb{R}^n)}+2^{(N+1)s}\|\dot{\Delta}_{N+1}u\|_{\mathrm{L}^p(\mathbb{R}^n)}\right)\\ &\lesssim_{p,n,s} \|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}.\end{aligned}$$

This shows that $\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}$ is finite, which ends the proof. ∎

One may use it to obtain interpolation inequalities,

Lemma 2.8 *Let $p_0,p_1\in(1,+\infty)$, $s_0,s_1\in\mathbb{R}$, we set*

$$\left(\frac{1}{p},s\right) := (1-\theta)\left(\frac{1}{p_0},s_0\right)+\theta\left(\frac{1}{p_1},s_1\right).$$

For all $u\in\mathcal{S}'_h(\mathbb{R}^n)$, we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p_0,p_1,s_0,s_1,n} \|u\|^{1-\theta}_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)}\|u\|^{\theta}_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)}.$$

Proof. — For $u\in\mathcal{S}'_h(\mathbb{R}^n)$, as a direct consequence of Hölder's inequality, we have

$$\left(\sum_{j\in\mathbb{Z}}|2^{js}\dot{\Delta}_j u|^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j\in\mathbb{Z}}|2^{js_0}\dot{\Delta}_j u|^2\right)^{\frac{1-\theta}{2}}\left(\sum_{j\in\mathbb{Z}}|2^{js_1}\dot{\Delta}_j u|^2\right)^{\frac{\theta}{2}}.$$

Thus, one may take the L^p-norm of above inequality, and use again Hölder's inequality, so that

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \leqslant \left\| \|(2^{js_0}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\|^{1-\theta}_{\ell^2(\mathbb{Z})} \|(2^{js_1}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\|^{\theta}_{\ell^2(\mathbb{Z})} \right\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|u\|^{1-\theta}_{\dot{\mathrm{F}}^{s_0}_{p_0,2}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{\mathrm{F}}^{s_1}_{p_1,2}(\mathbb{R}^n)}. \qquad \blacksquare$$

Lemma 2.9 *Let* $p_j \in (1,+\infty)$, $s_j \in \mathbb{R}$, *for* $j \in \{0,1\}$. *If* $(\mathcal{C}_{s_0,p_0})$ *is satisfied then* $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$ *is a Banach space for which* $\mathcal{S}_0(\mathbb{R}^n)$ *is dense in it.*

Proof. — The completeness is straightforward. Concerning the claim about density, we follow the proof of [BCD11, Proposition 2.27] with minor modifications, in order to adapt it to our setting.

For $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, and fixed $\varepsilon > 0$, for $k \in \{0,1\}$ there exists $N \in \mathbb{N}$ such that for all $\tilde{N} \geqslant N$

$$\|u - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} < \varepsilon.$$

Here, for any $K \in \mathbb{N}$,

$$u_K := \sum_{|j|\leqslant K} \dot{\Delta}_j u.$$

For $M \in [\![\tilde{N}+1, +\infty[\![$, $R > 0$, provided $\Theta \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, real valued, supported in $B(0,2)$, such that $\Theta_{|_{B(0,1)}} = 1$, and $\Theta_R := \Theta(\cdot/R)$, we introduce

$$u^R_{\tilde{N},M} := (\mathrm{I} - \dot{S}_{-M})[\Theta_R u_{\tilde{N}}].$$

Since $\dot{\Delta}_k u_{\tilde{N}} = 0$, $k \leqslant -M-1$, we have $\dot{S}_{-M} u_{\tilde{N}} = 0$, then

$$u^R_{\tilde{N},M} - u_{\tilde{N}} = (\mathrm{I} - \dot{S}_{-M})[(\Theta_R - 1)u_{\tilde{N}}].$$

If one sets $m_k := \max(0, \lfloor s_k \rfloor + 2)$, since $0 \notin \operatorname{supp} \mathcal{F}(u^R_{\tilde{N},M} - u_{\tilde{N}})$ by construction, we apply [BL76, Theorem 6.3.2] and decreasing embedding of inhomogeneous Sobolev spaces to deduce

$$\begin{aligned}
\|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} &\lesssim_{M,s_k,p_k} \|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\mathrm{H}^{s_k,p_k}(\mathbb{R}^n)} \\
&\lesssim_{M,s_k,p_k} \|(\mathrm{I} - \dot{S}_{-M})[(\Theta_R - 1)u_{\tilde{N}}]\|_{\mathrm{H}^{m_k,p_k}(\mathbb{R}^n)} \\
&\lesssim_{M,s_k,p_k} \|[(\Theta_R - 1)u_{\tilde{N}}]\|_{\mathrm{H}^{m_k,p_k}(\mathbb{R}^n)}.
\end{aligned}$$

Since one may check that $u_{\tilde{N}} \in \mathrm{H}^{m_k,p_k}(\mathbb{R}^n)$ for $k \in \{0,1\}$, by dominated convergence theorem it follows that

$$\|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} \xrightarrow[R\to+\infty]{} 0.$$

Thus, for $R > 0$ big enough, we have for $k \in \{0,1\}$

$$\|u - u^R_{\tilde{N},M}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} < 2\varepsilon.$$

The proof ends here since $u^R_{\tilde{N},M} \in \mathcal{S}_0(\mathbb{R}^n)$. ■

We recall also the usual interpolation properties,

$$\begin{aligned}
&[\mathrm{H}^{s_0,p_0}(\mathbb{R}^n), \mathrm{H}^{s_1,p_1}(\mathbb{R}^n)]_\theta = \mathrm{H}^{s,p_\theta}(\mathbb{R}^n), &&(\mathrm{B}^{s_0}_{p,q_0}(\mathbb{R}^n), \mathrm{B}^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} = \mathrm{B}^s_{p,q}(\mathbb{R}^n), \\
&(\mathrm{H}^{s_0,p}(\mathbb{R}^n), \mathrm{H}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = \mathrm{B}^s_{p,q}(\mathbb{R}^n), &&[\mathrm{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \mathrm{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\theta = \mathrm{B}^s_{p_\theta,q_\theta}(\mathbb{R}^n),
\end{aligned}$$

whenever $(p_0,q_0), (p_1,q_1), (p,q) \in [1,+\infty]^2$ ($p \neq 1, +\infty$, when dealing with Sobolev (Riesz potential) spaces), $\theta \in (0,1)$, $s_0 \neq s_1$ two real numbers, such that

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right),$$

see [BL76, Theorem 6.4.5]. A similar statement is available for our homogeneous function spaces.

Proposition 2.10 *Let* $(p_0, p_1, p, q, q_0, q_1) \in (1,+\infty)^3 \times [1,+\infty]^3$, $s_0, s_1 \in \mathbb{R}$, *such that* $s_0 \neq s_1$, *and let*

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

Assuming $(\mathcal{C}_{s_0,p})$ *(resp.* $(\mathcal{C}_{s_0,p,q_0})$*), we get the following*

$$(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = (\dot{\mathrm{B}}^{s_0}_{p,q_0}(\mathbb{R}^n),\dot{\mathrm{B}}^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} = \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n). \tag{2.5}$$

If moreover $(\mathcal{C}_{s_0,p_0})$ *and* $(\mathcal{C}_{s_1,p_1})$ *are true then also is* $(\mathcal{C}_{s,p_\theta})$ *and*

$$[\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)]_{\theta} = \dot{\mathrm{H}}^{s,p_\theta}(\mathbb{R}^n), \tag{2.6}$$

and similarly if $(\mathcal{C}_{s_0,p_0,q_0})$ *and* $(\mathcal{C}_{s_1,p_1,q_1})$ *are satisfied then* $(\mathcal{C}_{s,p_\theta,q_\theta})$ *is also satisfied and*

$$[\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n),\dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_{\theta} = \dot{\mathrm{B}}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n). \tag{2.7}$$

Proof. — **Step 1:** Let us deal with the real interpolation identity (2.5). Let us consider first the case of Sobolev spaces, with $u\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. For $(a,b)\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, such that $u=a+b$, by Lemma 2.7 we have

$$(\dot\Delta_j u)_{j\in\mathbb{Z}} = (\dot\Delta_j a)_{j\in\mathbb{Z}} + (\dot\Delta_j b)_{j\in\mathbb{Z}} \in \mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))+\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})).$$

Therefore, by the definition of the K-functional and Lemma 2.7, for $t>0$,

$$\begin{aligned}K(t,(\dot\Delta_j u)_{j\in\mathbb{Z}},\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))) &\leqslant \|a\|_{\dot{\mathrm{F}}^{s_0}_{p,2}(\mathbb{R}^n)}+t\|b\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)}\\ &\lesssim_{p,s_0,s_1,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)}+t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)}.\end{aligned}$$

We then take the infimum on an all such pairs (a,b),

$$K(t,(\dot\Delta_j u)_{j\in\mathbb{Z}},\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))) \lesssim_{p,s_0,s_1,n} K(t,u,\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)). \tag{2.8}$$

Now, we want to prove the reverse estimate. Since $(\dot\Delta_j u)_{j\in\mathbb{Z}}\in\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))+\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))$, let $(A,B)\in\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))\times\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))$ such that

$$(\dot\Delta_j u)_{j\in\mathbb{Z}} = A+B. \tag{2.9}$$

For $(w_j)_{j\in\mathbb{Z}}\subset\mathcal{S}'(\mathbb{R}^n)$, say, for simplicity, with finite support in the discrete variable, we define the map

$$\tilde\Sigma((w_j)_{j\in\mathbb{Z}}) := \sum_{j=-\infty}^{+\infty}\dot\Delta_j[w_{j-1}+w_j+w_{j+1}], \tag{2.10}$$

and satisfies for $v\in\mathcal{S}'_h(\mathbb{R}^n)$

$$\tilde\Sigma((\dot\Delta_j v)_{j\in\mathbb{Z}}) = v.$$

By Lemma 2.7 and [Gra14a, Proposition 6.1.4], one can check that $\tilde\Sigma\,:\,\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))\longrightarrow\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ is well defined and bounded since $(\mathcal{C}_{s_0,p})$ is satisfied. Now, we apply $\tilde\Sigma$ to (2.9) to deduce from $\tilde\Sigma(\dot\Delta_j u)_{j\in\mathbb{Z}}=u\in\mathcal{S}'_h(\mathbb{R}^n)$, and $\tilde\Sigma A\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\subset\mathcal{S}'_h(\mathbb{R}^n)$, that

$$\tilde\Sigma B = u-\tilde\Sigma A\in\mathcal{S}'_h(\mathbb{R}^n).$$

By the mean of [Gra14a, Proposition 6.1.4], we obtain

$$\|\tilde\Sigma B\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)} = \|(\dot\Delta_j\tilde\Sigma B)_{j\in\mathbb{Z}}\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))} \lesssim_{p,s_1,n} \|B\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}.$$

Hence, by Lemma 2.7, $\tilde\Sigma B$ is an element of $\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. Therefore by the definition of the K-functional, the boundedness properties of $\tilde\Sigma$, Lemma 2.7 and [Gra14a, Proposition 6.1.4], for $t>0$,

$$\begin{aligned}K(t,u,\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) &\leqslant \|\tilde\Sigma A\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)}+t\|\tilde\Sigma B\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)}\\ &\lesssim_{p,s_0,s_1,n} \|A\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))}+t\|B\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}.\end{aligned}$$

Thus, let us take the infimum on all such pairs (A,B), and invoke (2.8) to obtain for all $t>0$, and all $u\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$,

$$K(t,(\dot\Delta_j u)_{j\in\mathbb{Z}},\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))) \sim_{p,s_0,s_1,n} K(t,u,\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)). \tag{2.11}$$

We recall that [BL76, Theorems 5.6.1 & 3.5.3] and [Tri78, Theorem, Section 1.18.4] give, all together, the well known real interpolation identity

$$(\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})))_{\theta,q} = \ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n)). \tag{2.12}$$

Thus, up to multiply the estimate (2.11) by $t^{-\theta}$ and taking its L^q_*-norm, it can be turned into

$$\begin{aligned}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = \|(\dot\Delta_j u)_{j\in\mathbb{Z}}\|_{\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))} &\sim_{p,s_0,s_1,\theta,n} \|(\dot\Delta_j u)_{j\in\mathbb{Z}}\|_{(\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})))_{\theta,q}}\\ &\sim_{p,s_0,s_1,\theta,n} \|u\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q}}.\end{aligned}$$

Therefore (2.5) is proved.

Step 2: For $p\in(1,+\infty)$, $q\in[1,+\infty]$ and $s\in\mathbb{R}$ such that ($\mathcal{C}_{s,p,q}$) is satisfied, for $\tilde\Sigma$ introduced in (2.10), we want to show the boundedness of

$$\tilde\Sigma\,:\,\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))\longrightarrow \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n).$$

Let $(u_j)_{j\in\mathbb{Z}}\in\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))$ with finite support with respect to the discrete variable, by the real interpolation identity (2.12), it holds that, for some fixed $s_0<s<s_1$,

$$(u_j)_{j\in\mathbb{Z}}\in \mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))+\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})).$$

Let $(a,b)\in \mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))\times\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))$, such that $(u_j)_{j\in\mathbb{Z}}=a+b$. Up to restrict a and b to the discrete support of $(u_j)_{j\in\mathbb{Z}}$, denoting those restriction by respectively $\tilde a$ and $\tilde b$, we obtain

$$(u_j)_{j\in\mathbb{Z}}=\tilde a+\tilde b.$$

Therefore, by finite support of $\tilde a$ and $\tilde b$ in the discrete variable and by Lemma 2.7,

$$\tilde\Sigma\tilde a\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\text{ and }\tilde\Sigma\tilde b\in\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n),$$

so that

$$\tilde\Sigma(u_j)_{j\in\mathbb{Z}}\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n).$$

Hence, by the definition of the K-functional, Lemma 2.7 and the boundedness properties of $\tilde\Sigma$,

$$\begin{aligned}K(t,\tilde\Sigma(u_j)_{j\in\mathbb{Z}},\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) &\lesssim_{p,s_0,s_1,n} \|\tilde\Sigma\tilde a\|_{\dot{\mathrm{F}}^{s_0}_{p,2}(\mathbb{R}^n)}+t\|\tilde\Sigma\tilde b\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)}\\ &\lesssim_{p,s_0,s_1,n} \|\tilde a\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))}+t\|\tilde b\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}\\ &\lesssim_{p,s_0,s_1,n} \|a\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))}+t\|b\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}.\end{aligned}$$

Now, one can take the infimum on all such pairs (a,b), to deduce that for all $t>0$,

$$K(t,\tilde\Sigma(u_j)_{j\in\mathbb{Z}},\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))\lesssim_{p,s_0,s_1,n} K(t,(u_j)_{j\in\mathbb{Z}},\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))).$$

Multiplying, this estimate by $t^{-\theta}$ then taking the L^q_*-norm yield, by the mean of (2.5) and (2.12), the estimate

$$\|\tilde\Sigma(u_j)_{j\in\mathbb{Z}}\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\lesssim_{p,s_0,s_1,\theta,n}\|(u_j)_{j\in\mathbb{Z}}\|_{\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))}.$$

Then the map $\tilde\Sigma$ extends uniquely as a bounded map whenever ($\mathcal{C}_{s,p,q}$) is satisfied and $q<+\infty$.

For the case $q=+\infty$ when ($\mathcal{C}_{s,p,q}$) is satisfied, *i.e.* when $s<n/p$ is satisfied, the result follows in fact directly from **Step 1**.

In fact, above manual real interpolation procedure was only needed to reach the endpoint couple $(\dot{\mathrm{B}}^{n/p}_{p,1}(\mathbb{R}^n),\ell^1_{n/p}(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n)))$.

Step 3: For the real interpolation identity (2.5) in the case of Besov spaces, by the previous **Step 2**, the proof presented in **Step 1** is still valid if we replace $(\dot{\mathrm{H}}^{s_0,p},\dot{\mathrm{H}}^{s_1,p})$ and the condition ($\mathcal{C}_{s_0,p}$) by $(\dot{\mathrm{B}}^{s_0}_{p,q_0},\dot{\mathrm{B}}^{s_1}_{p,q_1})$ with the condition ($\mathcal{C}_{s_0,p,q_0}$).

Step 4: As in the proof of [BL76, Theorem 6.4.5], being aware of [BL76, Definition 6.4.1], we can claim, thanks to previous steps, that

- thanks to its definition, for all $s\in\mathbb{R}$, $p\in(1,+\infty)$, $q\in[1,+\infty]$, when ($\mathcal{C}_{s,p,q}$) is satisfied, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is a retraction of $\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))$ on $\mathcal{S}'_h(\mathbb{R}^n)$ through the homogeneous Littlewood-Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$, and projection map $\tilde\Sigma$;
- similarly, due to Lemma 2.7, for all $s\in\mathbb{R}$, $p\in(1,+\infty)$, when ($\mathcal{C}_{s,p}$) is satisfied $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a retraction of $\mathrm{L}^p(\mathbb{R}^n,\ell^2_s(\mathbb{Z}))$ on $\mathcal{S}'_h(\mathbb{R}^n)$ through the homogeneous Littlewood-Paley decomposition $(\dot\Delta_j)_{j\in\mathbb{Z}}$, and projection map $\tilde\Sigma$.

Thus, one may apply [BL76, Theorem 6.4.2], with [BL76, Theorem 5.6.3] for complex interpolation of Besov spaces and [Tri78, Theorem, Section 1.18.4] for complex interpolation of Sobolev spaces, to obtain respectively (2.7) and (2.6).

The completeness assumption is necessary in the case of complex interpolation, since one can not provide in general an appropriate sense of holomorphic functions (then of the definition of complex interpolation spaces) in non-complete normed vector spaces. ■

Proposition 2.11 *For any* $s\in\mathbb{R}$, $p\in(1,+\infty)$,

$$\begin{cases} \dot{\mathrm{H}}^{s,p}\times\dot{\mathrm{H}}^{-s,p'} & \longrightarrow \mathbb{C} \\ (u,v) & \longmapsto \displaystyle\sum_{|j-j'|\leq 1}\big\langle\dot{\Delta}_j u,\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n} \end{cases}$$

defines a continuous bilinear functional on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$. *Denote by* $\mathcal{V}^{-s,p'}$ *the set of functions* $v\in\mathcal{S}(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$ *such that* $\|v\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)}\leqslant 1$. *If* $u\in\mathcal{S}'_h(\mathbb{R}^n)$, *then we have*

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}=\sup_{v\in\mathcal{V}^{-s,p'}}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|.$$

Moreover, if $(\mathcal{C}_{s,p})$ *is satisfied,* $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ *is reflexive and we have*

$$(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'=\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n). \tag{2.13}$$

Proof. — For simplicity, we will first work with the norm provided by the Lemma 2.7, by equivalence of norms, the result will remain true. Let $(u,v)\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, the $\mathrm{L}^p(\ell^2)$-$\mathrm{L}^{p'}(\ell^2)$ Hölder's inequality gives,

$$\begin{aligned}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big| &\leqslant \|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}\Big\|\big\|(2^{-js}[\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}]v)_{j\in\mathbb{Z}}\big\|_{\ell^2(\mathbb{Z})}\Big\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}\\ &\leqslant (2^{|s|+1}+1)\,\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}\,\|v\|_{\dot{\mathrm{F}}^{-s}_{p',2}(\mathbb{R}^n)}\,.\end{aligned}$$

Now, we know that it is a well defined quantity, we can compute

$$\begin{aligned}\big\langle u,v\big\rangle_{\mathbb{R}^n} &= \sum_{|j-j'|\leq 1}\big\langle\dot{\Delta}_j u,\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n}\\ &=\sum_{|j-j'|\leqslant 1}\big\langle(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u,(-\Delta)^{-\frac{s}{2}}\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n}\\ &=\big\langle(-\Delta)^{\frac{s}{2}}u,(-\Delta)^{-\frac{s}{2}}v\big\rangle_{\mathbb{R}^n}\,.\end{aligned}$$

Hence, Hölder's inequality gives

$$\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|\leqslant\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\|v\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)},$$

which can be turned effortless into

$$\sup_{v\in\mathcal{V}^{-s,p'}}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|\leqslant\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\,.$$

This also proves the continuous embedding $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\hookrightarrow(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$. For the reverse inequality, but not the reverse embedding, from $\mathrm{L}^p-\mathrm{L}^{p'}$ duality, by density of $\mathcal{S}_0(\mathbb{R}^n)$, we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}=\sup_{\substack{v\in\mathcal{S}_0(\mathbb{R}^n),\\ \|v\|_{\mathrm{L}^{p'}}\leqslant 1}}\big|\big\langle(-\Delta)^{\frac{s}{2}}u,v\big\rangle_{\mathbb{R}^n}\big|=\sup_{\substack{w\in\mathcal{S}_0(\mathbb{R}^n),\\ \|w\|_{\dot{\mathrm{H}}^{-s,p'}}\leqslant 1}}\big|\big\langle u,w\big\rangle_{\mathbb{R}^n}\big|\leqslant\sup_{v\in\mathcal{V}^{-s,p'}}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|.$$

In particular, the embedding $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\hookrightarrow(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$ always holds and is isometric.

Now, assume that $(\mathcal{C}_{s,p})$ holds. We recall that Remark 2.6 yields the reflexivity of $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. Let $\tilde{U}\in(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$, we have

$$\big|\big\langle\tilde{U},(-\Delta)^{\frac{s}{2}}v\big\rangle\big|\leqslant\|\tilde{U}\|_{(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'}\|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)},\quad v\in\mathcal{S}_0(\mathbb{R}^n).$$

Since the space $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\mathrm{L}^{p'}(\mathbb{R}^n)$, we deduce there exists a unique function $w\in\mathrm{L}^p(\mathbb{R}^n)$

such that,

$$\langle \tilde{U}, v\rangle = \langle w, (-\Delta)^{-\frac{s}{2}} v\rangle_{\mathbb{R}^n}, \quad v \in \mathcal{S}(\mathbb{R}^n).$$

Thus $u := (-\Delta)^{-\frac{s}{2}} w \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ by Corollary 2.5, and yields that $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \hookrightarrow (\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$ is surjective. ■

Proposition 2.12 *For any $s \in \mathbb{R}$, $p \in (1,+\infty)$, $q \in [1,+\infty]$,*

$$\begin{cases} \dot{\mathrm{B}}^s_{p,q} \times \dot{\mathrm{B}}^{-s}_{p',q'} & \longrightarrow \mathbb{C} \\ \quad (u,v) & \longmapsto \sum\limits_{|j-j'|\leqslant 1} \langle \dot{\Delta}_j u, \dot{\Delta}_{j'} v\rangle_{\mathbb{R}^n} \end{cases}$$

defines a continuous bilinear functional on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n) \times \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$. Denote by $\mathcal{Q}^{-s}_{p',q'}$ the set of functions $v \in \mathcal{S}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ such that $\|v\|_{\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)} \leqslant 1$. If $u \in \mathcal{S}'_h(\mathbb{R}^n)$, then we have

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{p,s,n} \sup_{v \in \mathcal{Q}^{-s}_{p',q'}} |\langle u, v\rangle_{\mathbb{R}^n}|.$$

Moreover, if $-n/p' < s < n/p$ is satisfied and $q \in (1,+\infty]$ then

$$(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n))' = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n) \text{ and } (\dot{\mathcal{B}}^{-s}_{p',\infty}(\mathbb{R}^n))' = \dot{\mathrm{B}}^s_{p,1}(\mathbb{R}^n). \tag{2.14}$$

The space $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is reflexive whenever both ($\mathcal{C}_{s,p,q}$) and $q \neq 1, +\infty$ are satisfied.

Proof. — The first part of the claim is just [BCD11, Proposition 2.29]. The claimed part about reflexivity and duality follows directly from the application of [BL76, Theorem 3.7.1] and of Propositions 2.10 and 2.11. ■

We recall that Besov spaces satisfy usual Sobolev-Lebesgue spaces embeddings, say,

Proposition 2.13 ([BCD11, Proposition 2.39]) *Let $p,q \in [1,+\infty]$, $s \in (0,n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

The following estimates hold

$$\begin{aligned}
\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} &\lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n)}, \ \forall u \in \dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n),\ r \in [1,q],\\
\|u\|_{\dot{\mathrm{B}}^{-s}_{q,r}(\mathbb{R}^n)} &\lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n),\ r \in [q,+\infty],\\
\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} &\lesssim_{n,s,p} \|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n)}, \ \forall u \in \dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n),\ r \in [1,\min(2,p)],\\
\|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n)} &\lesssim_{n,s,p} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n),\ r \in [\max(2,p),+\infty].
\end{aligned}$$

Moreover, if p is finite, we also have $\dot{\mathrm{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_0(\mathbb{R}^n)$ and, each embedding is dense whenever p,q and r are finite.

We also have a Sobolev-Besov multiplier result, which is useful for the construction of homogeneous Sobolev and Besov space on domains. The first presentation of this result in the inhomogeneous setting is due to Strichartz [Str67, Chapter II, Corollary 3.7], one may all so check [JK95, Proposition 3.5].

Proposition 2.14 *For all $p \in (1,+\infty)$, for all $s \in [0,\frac{1}{p})$, for all $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, we have $\mathbb{1}_{\mathbb{R}^n_+} u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$ with estimate*

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}. \tag{2.15}$$

We are going to use it to prove,

Proposition 2.15 *For all $p \in (1,+\infty)$, $q \in [1,+\infty]$, for all $s \in (-1+\frac{1}{p},\frac{1}{p})$, for all $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ (resp. $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$),*

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \quad \textit{(resp. } \|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \textit{).}$$

The same results still holds with (H,B) *instead of* $(\dot{\mathrm{H}},\dot{\mathrm{B}})$.

Proof. — We start from the result stated in the inhomogeneous case Proposition 2.14, which states the following in the case of the upper half-space, for all $p\in(1,+\infty)$, for all $s\in[0,\frac{1}{p})$, for all $u\in\mathrm{H}^{s,p}(\mathbb{R}^n)$

$$\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n}\|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)},$$

which becomes under equivalence of norms,

$$\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\mathrm{L}^p(\mathbb{R}^n)}+\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n}\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}+\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Plugging $u_\lambda:=u(\lambda\cdot)$ in above inequality, provided λ is a positive real number, since $\mathbb{1}_{\mathbb{R}^n_+}(\lambda\cdot)u_\lambda=\mathbb{1}_{\mathbb{R}^n_+}u_\lambda$, we obtain that

$$\lambda^{-\frac{n}{p}}\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\mathrm{L}^p(\mathbb{R}^n)}+\lambda^{s-\frac{n}{p}}\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n}\lambda^{-\frac{n}{p}}\|u\|_{\mathrm{L}^p(\mathbb{R}^n)}+\lambda^{s-\frac{n}{p}}\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Thus one may divide by $\lambda^{s-\frac{n}{p}}$, and then letting λ grow to infinity, we have

$$\|\mathbb{1}_{\mathbb{R}^n_+}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n}\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)},$$

so the result follows by density argument.

The result for $s\in(-1+\frac{1}{p},0)$ is a consequence of duality and density using the duality bracket defined on $\mathcal{S}_0(\mathbb{R}^n)\times\mathcal{S}_0(\mathbb{R}^n)$.

The Besov space case follows by real interpolation. ■

2.2 Function spaces on $\mathbb{R}^n_+$

Let $s\in\mathbb{R}$, $p\in(1,+\infty)$, $q\in[1,+\infty]$. Then for any $\mathrm{X}\in\{\mathrm{B}^s_{p,q},\dot{\mathrm{B}}^s_{p,q},\mathrm{H}^{s,p},\dot{\mathrm{H}}^{s,p}\}$, and we define

$$\mathrm{X}(\mathbb{R}^n_+):=\mathrm{X}(\mathbb{R}^n)_{|_{\mathbb{R}^n_+}},$$

with the usual quotient norm $\|u\|_{\mathrm{X}(\mathbb{R}^n_+)}:=\inf\limits_{\substack{\tilde{u}\in\mathrm{X}(\mathbb{R}^n),\\ \tilde{u}_{|_{\mathbb{R}^n_+}}=u.}}\|\tilde{u}\|_{\mathrm{X}(\mathbb{R}^n)}$. A direct consequence of the definition of those spaces is the density of $\mathcal{S}_0(\overline{\mathbb{R}^n_+})\subset\mathcal{S}(\overline{\mathbb{R}^n_+})$ in each of them, and also, and the completeness and reflexivity when their counterpart on $\mathbb{R}^n$ also are. We can also define,

$$\mathrm{X}_0(\mathbb{R}^n_+):=\left\{\,u\in\mathrm{X}(\mathbb{R}^n)\,\middle|\,\operatorname{supp}u\subset\overline{\mathbb{R}^n_+}\right\},$$

with natural induced norm $\|u\|_{\mathrm{X}_0(\mathbb{R}^n_+)}:=\|u\|_{\mathrm{X}(\mathbb{R}^n)}$. We always have the canonical continuous injection,

$$\mathrm{X}_0(\mathbb{R}^n_+)\hookrightarrow\mathrm{X}(\mathbb{R}^n_+).$$

If X and Y are different function spaces

- if one has continuous embedding

$$\mathrm{Y}(\mathbb{R}^n)\hookrightarrow\mathrm{X}(\mathbb{R}^n).$$

  a direct consequence from the definition is

$$\mathrm{Y}(\mathbb{R}^n_+)\hookrightarrow\mathrm{X}(\mathbb{R}^n_+),$$

  and similarly with X_0 and Y_0.

- We write $[\mathrm{X}\cap\mathrm{Y}](\mathbb{R}^n_+)$ the restriction of $\mathrm{X}(\mathbb{R}^n)\cap\mathrm{Y}(\mathbb{R}^n)$ to $\mathbb{R}^n_+$, in general there is nothing to ensure more than

$$[\mathrm{X}\cap\mathrm{Y}](\mathbb{R}^n_+)\hookrightarrow\mathrm{X}(\mathbb{R}^n_+)\cap\mathrm{Y}(\mathbb{R}^n_+).$$

Results corresponding to those obtained for the whole space $\mathbb{R}^n$ in previous section are usually carried over by the existence of an appropriate extension operator

$$\mathcal{E}\,:\,\mathcal{S}'(\mathbb{R}^n_+)\longrightarrow\mathcal{S}'(\mathbb{R}^n),$$

bounded from $\mathrm{X}(\mathbb{R}^n_+)$ to $\mathrm{X}(\mathbb{R}^n)$.

2.2.1 Quick overview of inhomogeneous function spaces on $\mathbb{R}^n_+$

For inhomogeneous spaces on special Lipschitz domains (in particular on $\mathbb{R}^n_+$), an approach was done by Stein in [Ste70, Chapter VI], for Sobolev spaces with non-negative index, and Besov spaces of positive index of regularity (this follows by real interpolation). A full and definitive result for the inhomogeneous case on Lipschitz domains, and even in a more general case (allowing p, q to be less than 1 considering the whole Besov and Triebel-Lizorkin scales), was given by Rychkov in [Ryc99] where the extension operator is known to be universal and to cover even negative regularity index.

The extension operator provided by Rychkov can be used to prove, thanks to [BL76, Theorem 6.4.2], if $(\mathfrak{h},\mathfrak{b}) \in \{(\mathrm{H},\mathrm{B}),(\mathrm{H}_0,\mathrm{B}_{\cdot,\cdot,0})\}$,

$$[\mathfrak{h}^{s_0,p_0}(\mathbb{R}^n_+),\mathfrak{h}^{s_1,p_1}(\mathbb{R}^n_+)]_\theta = \mathfrak{h}^{s,p_\theta}(\mathbb{R}^n_+), \qquad (\mathfrak{b}^{s_0}_{p,q_0}(\mathbb{R}^n_+),\mathfrak{b}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q} = \mathfrak{b}^{s}_{p,q}(\mathbb{R}^n_+), \tag{2.16}$$

$$(\mathfrak{h}^{s_0,p}(\mathbb{R}^n_+),\mathfrak{h}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \mathfrak{b}^{s}_{p,q}(\mathbb{R}^n_+), \qquad [\mathfrak{b}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+),\mathfrak{b}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)]_\theta = \mathfrak{b}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n_+), \tag{2.17}$$

whenever $(p_0,q_0),(p_1,q_1),(p,q) \in [1,+\infty]^2$ ($p \neq 1,+\infty$, when dealing with Sobolev (Bessel potential) spaces), $\theta \in (0,1)$, $s_0 \neq s_1$ two real numbers, such that

$$\left(s,\frac{1}{p_\theta},\frac{1}{q_\theta}\right) := (1-\theta)\left(s_0,\frac{1}{p_0},\frac{1}{q_0}\right) + \theta\left(s_1,\frac{1}{p_1},\frac{1}{q_1}\right).$$

A nice property is that the description of the boundary yields the following density results, for all $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in \mathbb{R}$,

$$\mathrm{H}^{s,p}_0(\mathbb{R}^n_+) = \overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}}, \quad \text{and} \quad \mathrm{B}^s_{p,q,0}(\mathbb{R}^n_+) = \overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)}}. \tag{2.18}$$

One may check [JK95, Section 2] for the treatment of Sobolev spaces case, the Besov spaces case follows by interpolation argument, see [BL76, Theorem 3.4.2]. As a direct consequence, one has from [JK95, Proposition 2.9] and [BL76, Theorem 3.7.1], that for all $s \in \mathbb{R}$, $p \in (1,+\infty)$, $q \in [1,+\infty)$,

$$(\mathrm{H}^{s,p}(\mathbb{R}^n_+))' = \mathrm{H}^{-s,p'}_0(\mathbb{R}^n_+), \ (\mathrm{B}^s_{p,q}(\mathbb{R}^n_+))' = \mathrm{B}^{-s}_{p',q',0}(\mathbb{R}^n_+), \tag{2.19}$$

$$(\mathrm{B}^s_{p,q,0}(\mathbb{R}^n_+))' = \mathrm{B}^{-s}_{p',q'}(\mathbb{R}^n_+). \tag{2.20}$$

And finally, thanks to the inhomogeneous version of Proposition 2.15, we also have a particular case of equality of Sobolev spaces, with equivalent norms, for all $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+\frac{1}{p},\frac{1}{p})$,

$$\mathrm{H}^{s,p}(\mathbb{R}^n_+) = \mathrm{H}^{s,p}_0(\mathbb{R}^n_+), \ \mathrm{B}^s_{p,q}(\mathbb{R}^n_+) = \mathrm{B}^s_{p,q,0}(\mathbb{R}^n_+). \tag{2.21}$$

The interested reader may also found an explicit and way more general (and still valid, for the most part of it, in the case of the half-space) treatment for bounded Lipschitz domains in [KMM07], where the Triebel-Lizorkin scale, including Hardy spaces, and other endpoint function spaces are also treated.

All the results presented above will be used without being mentioned and are assumed to be well known to the reader.

2.2.2 Homogeneous function spaces on $\mathbb{R}^n_+$

One may expect to recover similar results for the scale of homogeneous Sobolev and Besov as the one mentioned in the subsection 2.2.1. However, due to the setting involving the use of $\mathcal{S}'_h(\mathbb{R}^n)$, we have a lack of completeness so that one can no longer use complex interpolation theory and density argument one the whole scale to provide boundedness of linear operators. A first approach we could have in mind is that one would expect Rychkov's extension operator to preserve $\mathcal{S}'_h$, say $\mathcal{E}(\mathcal{S}'_h(\mathbb{R}^n_+)) \subset \mathcal{S}'_h(\mathbb{R}^n)$ with ***homogeneous*** estimates, which is not known yet.

However, if we consider a more naive extension operator like by reflection around the boundary, as in [DHMT21, Chapter 3], a certain amount of results remains true, up to consider index $s > -1+\frac{1}{p}$, provided $p \in (1,+\infty)$. This is what we are going to achieve here: this subsection is devoted to proofs of usual results on homogeneous Sobolev and Besov spaces on $\mathbb{R}^n_+$. To be more clear, we are going to show via the previously mentioned extension-restriction operators, few duality arguments, and interpolation theory, that we still have:

- **Expected density results:**
For $p\in(1,+\infty)$, $q\in[1,+\infty)$, $s>-1+\frac{1}{p}$, when ($\mathcal{C}_{s,p,q}$) is satisfied,
$$\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)=\overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}},\quad\text{and}\quad \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)=\overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}}\ ;\tag{2.22}$$
- **Expected duality results:**
For all $p\in(1,+\infty)$, $q\in(1,+\infty]$, $s>-1+\frac{1}{p}$, when ($\mathcal{C}_{s,p,q}$) is satisfied,
$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+),\ (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'=\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+),\tag{2.23}$$
$$(\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+),\ (\dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_+))'=\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+).\tag{2.24}$$
- **Expected interpolation results:**
If $(\dot{\mathfrak{h}},\dot{\mathfrak{b}})\in\{(\dot{\mathrm{H}},\dot{\mathrm{B}}),(\dot{\mathrm{H}}_0,\dot{\mathrm{B}}_{\cdot,\cdot,0})\}$, with $(p_0,q_0),(p_1,q_1),(p,q)\in[1,+\infty]^2$ ($p,p_j\neq 1,+\infty$ is assumed, when dealing with Sobolev (Riesz potential) spaces), $\theta\in(0,1)$, $s_j,s>-1+1/p_j$, $j\in\{0,1\}$, with $s>-1+1/p$, where s_0,s_1,s are three real numbers, so that one can set
$$\left(s,\frac{1}{p_\theta},\frac{1}{q_\theta}\right):=(1-\theta)\left(s_0,\frac{1}{p_0},\frac{1}{q_0}\right)+\theta\left(s_1,\frac{1}{p_1},\frac{1}{q_1}\right),$$
such that either ($\mathcal{C}_{s,p_\theta,q_\theta}$) or ($\mathcal{C}_{s,p,q}$) is satisfied. Then, one has
$$[\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)]_\theta=\dot{\mathfrak{h}}^{s,p_\theta}(\mathbb{R}^n_+),\qquad (\dot{\mathfrak{b}}^{s_0}_{p,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q}=\dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+),\tag{2.25}$$
$$(\dot{\mathfrak{h}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}=\dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+),\qquad [\dot{\mathfrak{b}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)]_\theta=\dot{\mathfrak{b}}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n_+).\tag{2.26}$$

Note that, due to Proposition 2.15, we have already checked that following equalities of homogeneous Sobolev and Besov spaces remains true, with equivalent norms, for all $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(-1+\frac{1}{p},\frac{1}{p})$,
$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)=\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).\tag{2.27}$$
Some already existing density and boundedness results in Besov spaces presented here are already known, but redone here in a different manner giving some minor improvements with regard to [DHMT21, Chapter 3], allowing sometimes to deal sometimes with $s>-1+\frac{1}{p}$ or $q=+\infty$. Some other results, despite being well known in the construction of usual Sobolev and Besov spaces, are quite new due to the ambiant framework, this leads to some new proofs in a different spirit than the ones already available in the literature.

This subsection contains 3 subparts: the first one is about extension-restriction and density results for our homogeneous Sobolev spaces, from which for the second, we are going to build corresponding ones for Besov spaces, via some ersatz of real interpolation procedure. Both will be used to build the third subpart which concerns effective interpolation results for our homogeneous Sobolev and Besov spaces.

We start by proving, in a similar fashion to what has been already done in [DHMT21, Lemma 3.15, Proposition 3.19] for homogeneous Besov spaces, the boundedness of extension operators defined by higher order reflection principle but for homogeneous Sobolev spaces with fractional index of regularity.

Proposition 2.16 *For $m\in\mathbb{N}$, there exists a linear extension operator E, depending on m, such that for all $p\in(1,+\infty)$, $-1+\frac{1}{p}<s<m+1+\frac{1}{p}$, so that if either,*

- *$s\geqslant 0$ and $u\in\mathrm{H}^{s,p}(\mathbb{R}^n_+)$;*
- *$s\in(-1+\frac{1}{p},\frac{1}{p})$ and $u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$;*

we have
$$\mathrm{E}u_{|_{\mathbb{R}^n_+}}=u,$$

with estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\,.$$

In particular, $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ *uniquely extends as a bounded operator whenever* $(\mathcal{C}_{s,p})$ *is satisfied.*

Proof. — As in [DHMT21, Lemma 3.15], let us introduce the higher order reflexion operator E, defined for all measurable function $u : \mathbb{R}^n_+ \longrightarrow \mathbb{C}$ by

$$\mathrm{E}u(x) := \begin{cases} u(x) & \text{, if } x \in \mathbb{R}^n_+,\\ \sum_{j=0}^m \alpha_j u(x', -\frac{x_n}{j+1}) & \text{, if } x \in \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases}$$

where, as in [DHMT21, Lemma 3.15], $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$, and $(\alpha_j)_{j\in[\![0,m]\!]}$ is such that E maps C^m-functions on $\mathbb{R}^n_+$ to C^m-functions on $\mathbb{R}^n$, and by construction it also maps boundedly $\mathrm{H}^{k,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{k,p}(\mathbb{R}^n)$ for all $k \in [\![0, m+1]\!]$ then $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{s,p}(\mathbb{R}^n)$ for all $s \in [0, m+1]$ by complex interpolation.

Notice also that Proposition 2.15 and the formulation, given for $x \in \mathbb{R}^n$,

$$\mathrm{E}u(x) = [\mathbb{1}_{\mathbb{R}^n_+} u](x) + \sum_{j=0}^m \alpha_j [\mathbb{1}_{\mathbb{R}^n_+} u](x', -\tfrac{x_n}{j+1})$$

implies that $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded for all $s \in (-1+\frac{1}{p}, \frac{1}{p})$.

Now for $p \in (1, +\infty)$, $s \in [0, m+1+\frac{1}{p})$, $s - \frac{1}{p} \notin \mathbb{N}$, $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, $\mathrm{E} : \mathrm{H}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, we can whose $\ell \in \mathbb{N}$ such that $s - \ell \in (-1+\frac{1}{p}, \frac{1}{p})$ so that

$$\begin{aligned} \partial^\ell_{x_k} \mathrm{E}u &= \mathrm{E}[\partial^\ell_{x_\ell} u], \text{ provided } k \in [\![1, n-1]\!],\\ \partial^\ell_{x_n} \mathrm{E}u &= \mathrm{E}^{(\ell)} \partial^\ell_{x_n} u = \sum_{j=0}^m \alpha_j \left(\tfrac{-1}{j+1}\right)^\ell \partial^\ell_{x_n} u(x', -\tfrac{x_n}{j+1}). \end{aligned}$$

For the same reasons as in the beginning of the present proof, $\mathrm{E}^{(\ell)}$ maps $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{s,p}(\mathbb{R}^n)$ for all $s \in [0, m-\ell+1]$, and $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ to $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ for $s \in (-1+1/p, 1/p)$, thanks to Proposition 2.15.

From the fact that $\partial^\ell_{x_j} u \in \dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n_+)$, we deduce

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \sim_{\ell,p,n} \sum_{j=1}^{n-1} \|\partial^\ell_{x_j} \mathrm{E}u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)} + \|\mathrm{E}^{(\ell)}\partial^\ell_{x_n} u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)} \lesssim_{s,\ell,p,n,m} \sum_{j=1}^{n} \|\partial^\ell_{x_j} u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)}. \tag{2.28}$$

To be more synthetic, we have obtained

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,k,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$

so that $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded on subspace $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$, in particular it extends as a bounded linear operator on whole $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ when it is complete, i.e. $s < \frac{n}{p}$, this follows from the fact that $\mathcal{S}(\overline{\mathbb{R}^n_+}) \subset \mathrm{H}^{s,p}(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$.

It remains to cover cases when $s - \frac{1}{p} \in [\![0, m]\!]$. To do so, we want to reproduce above procedure, proving first that E (*resp.* $\mathrm{E}^{(\ell)}$, $\ell \in [\![1, m]\!]$) is bounded from $\dot{\mathrm{H}}^{\frac{1}{p},p}(\mathbb{R}^n_+)$ to $\dot{\mathrm{H}}^{\frac{1}{p},p}(\mathbb{R}^n)$, via some complex interpolation scheme.

Now let $p_0, p_1 \in (1, +\infty)$, $p_1 < n$, $\theta \in (0,1)$. Consider $u \in [\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta$. Let $f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+))$, such that $f(\theta) = u$, it follows from above considerations that $\mathrm{E}f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n))$, thus from Proposition 2.10,

$$\mathrm{E}f(\theta) \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n), \text{ where } \left(\theta, \frac{1}{p}\right) := (1-\theta)\left(0, \frac{1}{p_0}\right) + \theta\left(1, \frac{1}{p_1}\right).$$

So $u = \mathrm{E}f(\theta)_{|_{\mathbb{R}^n_+}} \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)$ with norm estimate

$$\|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)} \lesssim_{m_1,p,n} \|u\|_{[\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta}$$

which is a direct consequence of the definition of restriction space, the equivalence of the complex interpolation norm (2.6) from Proposition 2.10, the definition of the complex interpolation norm, and then of the boundedness of E on L^{p_0} and $\dot{\mathrm{H}}^{1,p_1}$. Now, if $u \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)$, by definition of restriction spaces there exists $U \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)$, such that

$$U_{|_{\mathbb{R}^n_+}} = u, \quad \text{and} \quad \frac{1}{2}\|U\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)} \leqslant \|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)} \leqslant \|U\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)}.$$

By Proposition 2.10, there exists $f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n))$ such that $f(\theta) = U$, we deduce $f(\cdot)_{|_{\mathbb{R}^n_+}} \in F(\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+))$, so $u = f(\theta)_{|_{\mathbb{R}^n_+}} \in [\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta$ with the following estimate which is a direct consequence from definitions of function spaces by restriction, and complex interpolation spaces,

$$\|u\|_{[\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta} \lesssim \|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)}.$$

Hence, homogeneous (Riesz potential) Sobolev spaces on the half-space are still a complex interpolation scale provided that $p \in (1,+\infty)$, $s \in [0,1]$, $(\mathcal{C}_{s,p})$ being satisfied, so the boundedness of $\mathrm{E} : \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+) \to \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)$ follows by interpolation.

In particular $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded for all $s \in (-1+\frac{1}{p}, \frac{1}{p}]$. Hence the result has been proved for $s - \frac{1}{p} = 0$. The same result is obtained for $\mathrm{E}^{(\ell)}$, provided $\ell \in [\![1,m]\!]$.

Now let $p \in (1,+\infty)$, $s - \frac{1}{p} \in [\![1,m]\!]$, for $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, $\nabla^\ell \mathrm{E}u \in \dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)$, $s - \ell = \frac{1}{p}$, so that, similarly as in (2.28),

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\ell} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Therefore, we have obtained the desired estimate and can conclude about the boundedness of E via density argument whenever $(\mathcal{C}_{s,p})$ is satisfied. ■

In the proof of Proposition 2.16, we used boundedness of derivatives, *i.e.* for all $p \in (1,+\infty)$, $s \in \mathbb{R}$, $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, $m \in \mathbb{N}$,

$$\|\nabla^m u\|_{\dot{\mathrm{H}}^{s-m,p}(\mathbb{R}^n_+)} \lesssim_{p,s,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}. \tag{2.29}$$

The above estimate is a direct consequence of definition of function spaces by restriction and can be turned into an equivalence under some additional assumptions.

Proposition 2.17 *Let* $p \in (1,+\infty)$, $k \in [\![1,+\infty[\![$, $s > k-1+\frac{1}{p}$, *for all* $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$,

$$\sum_{j=1}^{n} \|\partial^k_{x_j} u\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)} \sim_{s,k,p,n} \|\nabla^k u\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)} \sim_{s,k,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

In particular, $\|\nabla^k\cdot\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)}$ *and* $\sum_{j=1}^n \|\partial^k_{x_j}\cdot\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)}$ *provide equivalent norms on* $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, *whenever* $(\mathcal{C}_{s,p})$ *is satisfied.*

Proof. — Let us prove it for $k = 1$, the higher order case can be achieved in a similar manner. Consider $p \in (1,+\infty)$, $s > \frac{1}{p}$, for $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, where E is an extension operator provided by Proposition 2.16 (for some big enough $m \geqslant 1$), $\nabla \mathrm{E}u \in \dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)$, with $s - 1 > -1 + \frac{1}{p}$. We can write on ${\mathbb{R}^n_+}^c$

$$\partial_{x_\ell}\mathrm{E}u = \mathrm{E}[\partial_{x_\ell}u], \text{ provided } \ell \in [\![1,n-1]\!], \text{ and } \partial_{x_n}\mathrm{E}u = \sum_{j=0}^{m} \alpha_j \left(\tfrac{-1}{j+1}\right) \partial_{x_n} u(x', -\tfrac{x_n}{j+1}).$$

Hence, we can use definition of restriction space, apply Proposition 2.1, and boundedness of E, since m is large enough, to obtain,

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \leqslant \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla \mathrm{E}u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,m} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n_+)}.$$

Therefore by (2.29), the equivalence of norms on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ holds by density when $(\mathcal{C}_{s,p})$ is true. ■

The next proposition is about identifying intersection of homogeneous Sobolev spaces on $\mathbb{R}^n_+$, and give a dense subspace. As we can see later this will help for real interpolation.

Proposition 2.18 *Let $p_j \in (1,+\infty)$, $s_j > -1+\frac{1}{p_j}$, $j\in\{0,1\}$, if $(\mathcal{C}_{s_0,p_0})$ is satisfied then the following equality of vector spaces holds with equivalence of norms*

$$\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+) = [\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+).$$

In particular, $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$ is a Banach space which admits $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ as a dense subspace.

Proof. — Let $p\in(1,+\infty)$, $s_0,s_1\in\mathbb{R}$, such that $(\mathcal{C}_{s_0,p_0})$. By definition of restriction spaces and Lemma 2.9, $[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$ is complete and admits $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ as a dense subspace. The following continuous embedding also holds by definition,

$$[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)\hookrightarrow\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+).$$

Hence, it suffices to prove the reverse one. To do so, let us choose $\ell\in\mathbb{N}$ such that $(\mathcal{C}_{s_1-\ell,p_1})$ is satisfied, and $s_1-\ell > -1+\frac{1}{p_1}$, then choosing E from Proposition 2.16 with $m+1+\frac{1}{p_j} > s_j$, $j\in\{0,1\}$ (m big enough), for all $j\in[\![1,n]\!]$, and all $u\in\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$, $\mathrm{E}u$ makes sense in $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)$ then in $\mathcal{S}'_h(\mathbb{R}^n)$ and one may use an estimate similar to (2.28), to deduce

$$\sum_{k=1}^{n}\|\partial^{\ell}_{x_k}\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} = \sum_{k=1}^{n-1}\|\mathrm{E}\partial^{\ell}_{x_k}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} + \|\mathrm{E}^{(\ell)}\partial^{\ell}_{x_n}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} \lesssim^{p_1,n}_{s_1,m,\ell}\|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

The above operator $\mathrm{E}^{(\ell)}$ is given via the identity $\partial^{\ell}_{x_n}\mathrm{E} = \mathrm{E}^{(\ell)}\partial^{\ell}_{x_n}$. Hence, it follows that for all $u\in\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$,

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} + \sum_{k=1}^{n}\|\partial^{\ell}_{x_k}\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} \lesssim^{p_0,p_1,n}_{s_0,s_1,m,\ell}\|u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

In particular, since $\mathrm{E}u\in\mathcal{S}'_h(\mathbb{R}^n)$, and by uniqueness of representation of $\partial^{\ell}_{x_j}\mathrm{E}u$ in $\mathcal{S}'(\mathbb{R}^n)$, we deduce from Proposition 2.1 that $\mathrm{E}u\in\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$.

Thus $u\in[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$, and by definition of restriction spaces,

$$\|u\|_{[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)} \leqslant \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} + \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)} \lesssim^{p_0,p_1,n}_{s_0,s_1,m,\ell}\|u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

This proves the claim. ■

So one can deduce the following corollary which allows separate homogeneous estimates for intersection of homogeneous Sobolev spaces on $\mathbb{R}^n_+$. Since the estimates below are decoupled, it provides an ersatz of extension-restriction operators for homogeneous Sobolev spaces of higher order, thanks to the taken intersection yielding a complete space. For instance, this will be of use to circumvent the lack of completeness when we will want to (real-)interpolate between a "higher" order homogeneous Sobolev space, and one that is known to be complete.

Corollary 2.19 *Let $p_j\in(1,+\infty)$, $s_j > -1+\frac{1}{p_j}$, $j\in\{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied, consider $m\in\mathbb{N}$ such that $s_j < m+1+\frac{1}{p_j}$, and the extension operator E given by Proposition 2.16.*

Then for all $u\in\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$, we have $\mathrm{E}u\in\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)$, $j\in\{0,1\}$, with estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,m,n}\|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.$$

Previous Corollary 2.19 and the proof of Proposition 2.17 lead to

Corollary 2.20 *Let $p_j\in(1,+\infty)$, $m_j\in[\![1,+\infty[\![$, $s_j > m_j-1+\frac{1}{p_j}$, $j\in\{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. Then for all $u\in\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$,*

$$\sum_{k=1}^{n}\|\partial^{m_j}_{x_k}u\|_{\dot{\mathrm{H}}^{s_j-m_j,p}(\mathbb{R}^n_+)} \sim_{s_j,m_j,p_j,n}\|\nabla^{m_j}u\|_{\dot{\mathrm{H}}^{s_j-m_j,p_j}(\mathbb{R}^n_+)} \sim_{s_j,m_j,p_j,n}\|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.$$

Since one may also be interested into Sobolev spaces with 0-boundary condition, we introduce a projection operator that allows to deal with the interpolation property, and to recover, later on, some appropriate density results.

Lemma 2.21 *Let $p \in (1,+\infty)$, $s \in \mathbb{R}$, $m \in \mathbb{N}$, such that $-1+\frac{1}{p} < s < m+1+\frac{1}{p}$, then there exists a bounded projection $\mathcal{P}_0$, depending on m, such that it maps $\mathrm{H}^{s,p}(\mathbb{R}^n)$ to $\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$.*

If either

- *$s \geqslant 0$ and $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$;*
- *$s \in (-1+\frac{1}{p}, \frac{1}{p})$ and $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$;*

we have the estimate

$$\|\mathcal{P}_0 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,m,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

In particular, $\mathcal{P}_0$ extends as a bounded projection from $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ to $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p})$ is satisfied.

Proof. — Let $p \in (1,+\infty)$, $s > -1+\frac{1}{p}$, $m \in \mathbb{N}$, such that $s < m+1+\frac{1}{p}$. Then we consider the operator E given by Proposition 2.16, but we modify it into an operator E^-, for any measurable function $u : \mathbb{R}^n_- \longrightarrow \mathbb{C}$, we set for almost every $x \in \mathbb{R}^n$

$$\mathrm{E}^- u(x) := \begin{cases} u(x) & \text{, if } x \in \mathbb{R}^n_-, \\ \sum_{j=0}^{m} \alpha_j u(x', -\frac{x_n}{j+1}) & \text{, if } x \in \mathbb{R}^n \setminus \mathbb{R}^n_-. \end{cases}$$

Hence for any measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{C}$, we set for almost every $x \in \mathbb{R}^n$,

$$\mathcal{P}_0 u := u - \mathrm{E}^-[\mathbb{1}_{\mathbb{R}^n_-} u].$$

The fact that $\mathcal{P}_0^2 = \mathcal{P}_0$ is clear by definition, and we have $\mathcal{P}_0 \mathrm{H}^{s,p}(\mathbb{R}^n) \subset \mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$, and that ${\mathcal{P}_0}_{|_{\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)}} = \mathrm{I}$. Claimed boundedness properties follow from Proposition 2.15 and Proposition 2.16. ■

As well as the extension operator given by higher order reflection principle, the projection operator on "0-boundary condition" homogeneous Sobolev spaces satisfies homogeneous estimates on intersection spaces. The proof is a direct consequence of Proposition 2.18 and its formula introduced in the proof of Lemma 2.21.

Corollary 2.22 *Let $p_j \in (1,+\infty)$, $s_j > -1+\frac{1}{p_j}$, $j \in \{0,1\}$, $m \in \mathbb{N}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied and $s_j < m+1+\frac{1}{p_j}$, and consider the projection operator $\mathcal{P}_0$ given by Lemma 2.21.*

Then for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, we have $\mathcal{P}_0 u \in \dot{\mathrm{H}}^{s_j,p_j}_0(\mathbb{R}^n_+)$, $j \in \{0,1\}$, with estimate

$$\|\mathcal{P}_0 u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,m,p,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)}.$$

We still have Sobolev embeddings by definition of function spaces by restriction.

Proposition 2.23 *Let $p,q \in (1,+\infty)$, $s \in [0,n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

We have dense embeddings,

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}, \ \forall u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \tag{2.30}$$

$$\|u\|_{\dot{\mathrm{H}}^{-s,q}_0(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n_+). \tag{2.31}$$

Proof. — First let us recall the Hardy-Littlewood-Sobolev inequality from Proposition 2.3, which says that

$$\|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Hence, the embedding (2.31) is an obvious consequence plugging $1_{\mathbb{R}^n_+} u$ for $u \in \mathrm{L}^p(\mathbb{R}^n)$.

The embedding (2.30) is a direct consequence of 2.3 and function spaces defined by restriction. Indeed, for $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, we have for any extension $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \subset \mathrm{L}^q(\mathbb{R}^n)$ such that $u = U_{|_{\mathbb{R}^n_+}} \in \mathrm{L}^q(\mathbb{R}^n_+)$ the estimate

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \leqslant \|U\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{s,p,q,n} \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Looking at the infimum on all such U gives the result.

The density for the first embedding follows from the fact that $\mathcal{S}_0(\overline{\mathbb{R}^n_+}) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ is dense in $\mathrm{L}^q(\mathbb{R}^n_+)$. The density in the second case, follows from the canonical embedding,

$$\mathrm{L}^p(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{-s,q}_0(\mathbb{R}^n_+) \hookrightarrow \mathrm{H}^{-s,q}_0(\mathbb{R}^n_+),$$

which turn, by duality into embeddings,

$$\mathrm{H}^{s,q'}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{-s,q}_0(\mathbb{R}^n_+))' \hookrightarrow \mathrm{L}^{p'}(\mathbb{R}^n_+).$$

In particular, the following is a dense embedding

$$(\dot{\mathrm{H}}^{-s,q}_0(\mathbb{R}^n_+))' \hookrightarrow \mathrm{L}^{p'}(\mathbb{R}^n_+)$$

hence by reflexivity, the one below also is

$$\mathrm{L}^p(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{-s,q}_0(\mathbb{R}^n_+). \qquad \blacksquare$$

Now, all the ingredients are there in order to build the main usual density result for our 0-boundary conditions homogeneous Sobolev spaces.

Proposition 2.24 *For all $p \in (1,+\infty)$, $s \in (-\frac{n}{p'}, \frac{n}{p})$, the space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$.*

Proof. — First, let $s \in [0,n)$. Let $p \in (1,+\infty)$, such that $(\mathcal{C}_{s,p})$ is true, and consider $u \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$.

In particular, we have $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. Hence, there exists $(u_k)_{k\in\mathbb{N}} \subset \mathrm{H}^{s,p}(\mathbb{R}^n)$ such that

$$u_k \xrightarrow[k\to+\infty]{} u \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n).$$

Thus, it follows from Lemma 2.21, that $(\mathcal{P}_0 u_k)_{k\in\mathbb{N}} \subset \mathrm{H}^{s,p}_0(\mathbb{R}^n_+) \subset \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$ converge to $\mathcal{P}_0 u = u$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. For $\varepsilon > 0$, there exists some k_0, such that for all $k \geqslant k_0$, we have

$$\|u - \mathcal{P}_0 u_k\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

Now, we use density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$ to assert that there exists $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$ so that,

$$\|\mathcal{P}_0 u_k - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} \leqslant \|\mathcal{P}_0 u_k - w\|_{\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

We can conclude for the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$, since

$$\|u - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} \leqslant \|u - \mathcal{P}_0 u_k\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} + \|\mathcal{P}_0 u_k - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} < 2\varepsilon.$$

Now let us consider $s \in (0, \frac{n}{p'})$, $u \in \dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)$, applying Proposition 2.23, for $\varepsilon > 0$ there exists a function $v \in \mathrm{L}^q(\mathbb{R}^n_+)$, (with $\frac{1}{p} = \frac{1}{q} - \frac{s}{n}$) such that,

$$\|u - v\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

But recalling that $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\mathrm{L}^q(\mathbb{R}^n_+)$, there exists $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$ such that

$$\|v - w\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|v - w\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \varepsilon,$$

so the triangle inequality gives

$$\|u - w\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \varepsilon,$$

which conclude the proof since $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$. $\blacksquare$

Proposition 2.25 *Let $p_j \in (1,+\infty)$, $s_j \geqslant 0$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. The space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s_0,p_0}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}_0(\mathbb{R}^n_+)$.*

Proof. — It suffices to reproduce the first part of the proof of above Proposition 2.24 by the mean of Corollary 2.22. ∎

Corollary 2.26 *For all* $p \in (1,+\infty)$, $s \in (-1+\frac{1}{p}, \frac{1}{p})$,

$$\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+) = \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+).$$

In particular, $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ *is dense in* $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ *for same range of indices.*

Proof. — This is a direct consequence of the definition of restriction spaces and Proposition 2.15, the density result follows from Proposition 2.24. ∎

Proposition 2.27 *Let* $p \in (1,+\infty)$, $s \in (-\frac{n}{p'}, \frac{n}{p})$, *we have*

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+) \text{ and } (\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+).$$

Proof. — First, consider $s \in (-\frac{n}{p'}, \frac{n}{p})$, let $\Phi \in \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+) \subset \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, then using definition of restriction spaces, the following map defines a linear functional on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$,

$$u \longmapsto \big\langle \Phi, \tilde{u} \big\rangle_{\mathbb{R}^n},$$

where $\tilde{u}$ is any extension of u, and notice that the action of Φ does not depend on the choice of such extension of u. Indeed, if $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is another extension of u, we obtain that $w := U - \tilde{u} \in \dot{\mathrm{H}}^{s,p}_0(\overline{\mathbb{R}^n_+}^c)$. It follows from Proposition 2.24 that w is a strong limit in $\dot{\mathrm{H}}^{s,p}_0(\overline{\mathbb{R}^n_+}^c)$ of a sequence of functions $(w_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\overline{\mathbb{R}^n_+}^c)$ so that, passing to the limit, in the duality bracket, we obtain

$$\big\langle \Phi, U \big\rangle_{\mathbb{R}^n} - \big\langle \Phi, \tilde{u} \big\rangle_{\mathbb{R}^n} = \big\langle \Phi, w \big\rangle_{\mathbb{R}^n} = 0.$$

This gives a well defined continuous injective map

$$\begin{cases} \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+) & \longrightarrow (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' \\ \Phi & \longmapsto \ \big\langle \Phi, \tilde{\cdot} \big\rangle_{\mathbb{R}^n} \end{cases}. \tag{2.32}$$

Now, let $\Psi \in (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))'$, for all $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, since $1_{\mathbb{R}^n_+} u = u$, we may write,

$$\langle \Psi, u \rangle = \langle \Psi, 1_{\mathbb{R}^n_+} \widetilde{u} \rangle,$$

for any extension $\tilde{u} \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ of u, hence as a direct consequence of the definition of restriction space $1_{\mathbb{R}^n_+}\Psi \in (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, so $1_{\mathbb{R}^n_+}\Psi \in \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+)$. The following map is well defined continuous and injective

$$\begin{cases} (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' & \longrightarrow \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+) \\ \Psi & \longmapsto \ 1_{\mathbb{R}^n_+}\Psi \end{cases}. \tag{2.33}$$

Both maps (2.32) and (2.33) are even isometric and we obtain,

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+),$$

which was the first statement. The second statement follows from duality and reflexivity exchanging roles of involved exponents. ∎

The next result aim to carry over density in intersection spaces to transfer itself as a density result in their real interpolation spaces.

Corollary 2.28 *Let* $p \in (1,+\infty)$, $-n/p' < s_0 < s_1 < n/p$, *i.e. such that* $(\mathcal{C}_{-s_0,p'})$ *and* $(\mathcal{C}_{s_1,p})$ *are both satisfied.*

The space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ *is dense in* $\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)$.

Proof. — Let $p \in (1,+\infty)$, $-n/p' < s_0 < s_1 < n/p$. There are three subcases, $0 \leqslant s_0 < s_1$, $s_0 < 0 < s_1$, and $s_0 < s_1 \leqslant 0$.

The case $0 \leqslant s_0 < s_1$ follows the lines of Proposition 2.24 thanks to Corollary 2.22.

The case $s_0 < 0 < s_1$, can be done via duality argument as in Proposition 2.24 for the negative index of regularity. Let us consider $\frac{1}{q} = \frac{1}{p} - \frac{s_0}{n}$, the following embeddings are true

$$\mathrm{H}^{s_1-s_0,q}_0(\mathbb{R}^n_+) \hookrightarrow \mathrm{L}^q(\mathbb{R}^n_+) \cap \mathrm{H}^{s_1,p}_0(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+).$$

One may dualize it to deduce

$$\dot{\mathrm{H}}^{-s_1,p'}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))' \hookrightarrow \mathrm{H}^{s_0-s_1,q'}(\mathbb{R}^n_+).$$

We deduce that the last embedding is dense, since $(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))'$ contains $\dot{\mathrm{H}}^{-s_1,p'}(\mathbb{R}^n_+)$ via canonical embedding, so that by duality and reflexivity of all involved spaces, the following embedding is dense:

$$\mathrm{H}^{s_1-s_0,q}_0(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+).$$

Since $\mathrm{C}^\infty_c(\mathbb{R}^n_+) \hookrightarrow \mathrm{H}^{s_1-s_0,q}_0(\mathbb{R}^n_+)$ is dense, the result follows.

We end the proof claiming that the third case $s_0 < s_1 \leqslant 0$ can be done similarly via duality and reflexivity arguments. ■

We are done with properties of homogeneous Sobolev spaces. We continue with a real interpolation embedding lemma, that will allow us to transfer all nice properties, like boundedness of extension and projection operators, from homogeneous Sobolev spaces to homogeneous Besov spaces.

Lemma 2.29 *Let $(p,q,q_0,q_1) \in (1,+\infty) \times [1,+\infty]^3$, $s_0, s_1 \in \mathbb{R}$, such that $s_0 < s_1$, and set*

$$s := (1-\theta)s_0 + \theta s_1.$$

If $(\mathcal{C}_{s_0,p})$ is satisfied we have,

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}, \tag{2.34}$$

$$\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+) \hookleftarrow (\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q}. \tag{2.35}$$

Similarly if $(\mathcal{C}_{s_0,p,q_0})$ is satisfied, we also have

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{B}}^{s_0}_{p,q_0}(\mathbb{R}^n_+), \dot{\mathrm{B}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q}, \tag{2.36}$$

$$\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+) \hookleftarrow (\dot{\mathrm{B}}^{s_0}_{p,q_0,0}(\mathbb{R}^n_+), \dot{\mathrm{B}}^{s_1}_{p,q_1,0}(\mathbb{R}^n_+))_{\theta,q}. \tag{2.37}$$

Proof. — For embeddings (2.34) and (2.36), one may follow the first part of the proof of [DHMT21, Proposition 3.22].

The third embedding (2.35), (the fourth one (2.37) can be treated similarly) is straightforward since,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n).$$

By definition, $f \in (\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \subset \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)$, hence $\operatorname{supp} f \subset \overline{\mathbb{R}^n_+}$ and $f \in \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$. ■

As we mentioned, the above lemma can be used to prove boundedness of some operator on a sufficiently large range of indices on Besov spaces via some sort of interpolation method, without full information about exact description of the interpolation space, see below.

Corollary 2.30 *Let $p \in (1,+\infty), q \in [1,+\infty]$, $s > -1+\frac{1}{p}$, $m \in \mathbb{N}$, such that $s < m+1+\frac{1}{p}$. Let us consider the extension operator E (resp. $\mathcal{P}_0$) given by Proposition 2.16 (resp. Lemma 2.21).*

If either

- *$s > 0$ and $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$ (resp. $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n)$) ;*
- *$s \in (-1+\frac{1}{p}, \frac{1}{p})$ and $u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ (resp. $u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$) ;*

we have the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,m,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \text{ . (resp. } \|\mathcal{P}_0 u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,m,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \text{ .)}$$

In particular, E *(resp.* $\mathcal{P}_0$*) is a bounded operator from* $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ *to* $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ *(resp. from* $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ *to* $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$*) whenever* ($\mathcal{C}_{s,p,q}$) *is satisfied.*

Proof. — Let $p \in (1,+\infty), q \in [1,+\infty)$, $s > -1+\frac{1}{p}$, $m \in \mathbb{N}$, such that $s < m+1+\frac{1}{p}$. Without loss of generality, it suffices to prove the result for the operator E, since we have the identity $\mathcal{P}_0 = \mathrm{I} - \mathrm{E}^-[1_{\mathbb{R}^n_-}]$, as written in the proof of Lemma 2.21.

The boundedness of E on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ for $(p,q) \in (1,+\infty)\times[1,+\infty]$, $s \in (-1+\frac{1}{p}, \frac{1}{p})$ is again a direct consequence of Proposition 2.15.

It remains to prove boundedness for $s \geqslant \frac{1}{p}$. To do so, we proceed via a manual real interpolation scheme.

Let $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, $\theta \in (0,1)$ such that $\theta s_1 = s$, where $s_1 \in (s, m+1+\frac{1}{p})$. One has

$$u \in (\mathrm{L}^p(\mathbb{R}^n_+), \mathrm{H}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} \hookrightarrow (\mathrm{L}^p(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} \subset \mathrm{L}^p(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+).$$

Hence, for $a \in \mathrm{L}^p(\mathbb{R}^n_+)$, $b \in \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$ such that $f = a+b$, we can deduce that

$$b = u - a \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+) + \mathrm{L}^p(\mathbb{R}^n_+) \subset \mathrm{L}^p(\mathbb{R}^n_+),$$

so that $b \in \mathrm{L}^p(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+) = \mathrm{H}^{s_1,p}(\mathbb{R}^n_+)$ thanks to Proposition 2.18. Hence, $\mathrm{E}u = \mathrm{E}a + \mathrm{E}b$, with $\mathrm{E}a \in \mathrm{L}^p(\mathbb{R}^n_+)$, $\mathrm{E}b \in \mathrm{H}^{s_1,p}(\mathbb{R}^n_+)$, with homogeneous estimates provided by Proposition 2.16. Then $\mathrm{E}u_{|_{\mathbb{R}^n_+}} = u$, and we have estimates

$$K(t, \mathrm{E}u, \mathrm{L}^p(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \leqslant \|\mathrm{E}a\|_{\mathrm{L}^p(\mathbb{R}^n)} + t\|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{p,m,n} \|a\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

Hence, taking infimum on all such functions a and b, and multiplying by $t^{-\theta}$ leads to

$$t^{-\theta}K(t, \mathrm{E}u, \mathrm{L}^p(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \lesssim_{p,s,s_1,n} t^{-\theta}K(t, u, \mathrm{L}^p(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)),$$

so one may take the L^q_*-norm of above inequality and use (2.34) from Lemma 2.29 to deduce that

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

If $q < +\infty$, then $\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, so that the conclusion holds by density whenever ($\mathcal{C}_{s,p,q}$) is satisfied.

If $q = +\infty$, and ($\mathcal{C}_{s,p,q}$) is satisfied, necessarily $s < \frac{n}{p}$. We introduce $\mathcal{E} := \mathrm{E}[1_{\mathbb{R}^n_+}\cdot]$ which is bounded, thanks to the above step, seen as an operator

$$\mathcal{E} \,:\, \dot{\mathrm{B}}^{s_j}_{p,q_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{B}}^{s_j}_{p,q_j}(\mathbb{R}^n),$$

provided $s_0 < s < s_1 < \frac{n}{p}$, and $q_j \in [1,\infty)$, $j \in \{0,1\}$. Thus, by real interpolation argument, thanks to Proposition 2.10, for all $U \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)$, we have

$$\|\mathcal{E}U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}.$$

In particular, for all $u \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+)$, and all $U \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)$ such that $U_{|_{\mathbb{R}^n_+}} = u$, we have

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}.$$

Hence, taking the infimum on all such functions U gives the result when $q = +\infty$ and ($\mathcal{C}_{s,p,q}$) is satisfied. ■

Proposition 2.31 *Let* $p, q \in [1,+\infty]$, $s \in (0,n)$, *such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

We have the following estimates,

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \lesssim_{n,s,p,q,r} \|u\|_{\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+)},\ \forall u\in \dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+),\ r\in[1,q]$$
$$\|u\|_{\dot{\mathrm{B}}^{-s}_{q,r,0}(\mathbb{R}^n_+)} \lesssim_{n,s,p,q,r} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)},\ \forall u\in \mathrm{L}^p(\mathbb{R}^n_+),\ r\in[q,+\infty],$$
$$\|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \lesssim_{n,s,p} \|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n_+)},\ \forall u\in \dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n_+),\ r\in[1,\min(2,p)],$$
$$\|u\|_{\dot{\mathrm{B}}^{0}_{p,r,0}(\mathbb{R}^n_+)} \lesssim_{n,s,p,r} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)},\ \forall u\in \mathrm{L}^p(\mathbb{R}^n_+),\ r\in[\max(2,p),+\infty].$$

Moreover, we also have $\dot{\mathrm{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\hookrightarrow \mathrm{C}^0_0(\overline{\mathbb{R}^n_+})$, *whenever* p *is finite.*

Proof. — Each embedding is a direct consequence of the definition of each space and the corresponding ones on $\mathbb{R}^n$, see Proposition 2.13. ■

Lemma 2.32 *Let* $p\in(1,+\infty)$, $q\in[1,+\infty)$ *and* $s>0$. *The function space* $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ *is dense in* $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$ *whenever* ($\mathcal{C}_{s,p,q}$) *is satisfied.*

Proof. — As in the proof of Proposition 2.24, in the case of non negative index: by a successive approximations scheme, we use density of $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ in $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, to approximate functions in $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$. Then the boundedness of $\mathcal{P}_0$ on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, and the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\mathrm{B}^s_{p,q,0}(\mathbb{R}^n_+)$ yields the result. ■

Proposition 2.33 *Let* $(p_0,p_1,p,q)\in(1,+\infty)^3\times[1,+\infty]$, $s_0,s_1\in\mathbb{R}$, *such that* $s_0<s_1$, *let* $(\mathfrak{h},\mathfrak{b})\in\{(\mathrm{H},\mathrm{B}),(\mathrm{H}_0,\mathrm{B}_{\cdot,\cdot,0})\}$, *and set*

$$\left(s,\frac{1}{p_\theta}\right) := (1-\theta)\left(s_0,\frac{1}{p_0}\right)+\theta\left(s_1,\frac{1}{p_1}\right).$$

If either one of following assertions is satisfied,

(i) $q\in[1,+\infty)$, $s_j>-1+\frac{1}{p_j}$, $j\in\{0,1\}$;

(ii) $q\in[1,+\infty]$, $s_j>-1+\frac{1}{p_j}$, *and* ($\mathcal{C}_{s_j,p_j}$) *is satisfied,* $j\in\{0,1\}$;

If $p_0=p_1=p$ *and* ($\mathcal{C}_{s,p,q}$) *is satisfied, the following equality is true with equivalence of norms*

$$(\dot{\mathfrak{h}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathfrak{b}}^s_{p,q}(\mathbb{R}^n_+). \tag{2.38}$$

If ($\mathcal{C}_{s_0,p_0}$) *and* ($\mathcal{C}_{s_1,p_1}$) *are true then also is* ($\mathcal{C}_{s,p_\theta}$) *and*

$$[\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)]_\theta = \dot{\mathfrak{h}}^{s,p_\theta}(\mathbb{R}^n_+). \tag{2.39}$$

Proof. — We start noticing that (2.39) only makes sense under assertion *(ii)*.

Step 1: We prove first (2.39) and (2.38) under assertion *(ii)*.

It suffices to assert that $\{\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)\}$ is a retraction of $\{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)\}$, thanks to [BL76, Theorem 6.4.2]. Indeed, both retractions are given by

$$\mathrm{E}\,:\,\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\text{ and }\mathrm{R}_{\mathbb{R}^n_+}\,:\,\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+),$$
$$\iota\,:\,\dot{\mathrm{H}}^{s_j,p_j}_0(\mathbb{R}^n_+)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\text{ and }\ \mathcal{P}_0\,:\,\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}_0(\mathbb{R}^n_+).$$

Here, $\mathrm{R}_{\mathbb{R}^n_+}$ and ι stand respectively for the restriction and the canonical injection operator. Boundedness and range of E and $\mathcal{P}_0$ provided by Lemma 2.21 and Corollary 2.30 lead to (2.39) and (2.38) under assertion *(ii)*.

Step 2: We prove (2.38) under assertion *(i)*.

Step 2.1: $(\mathfrak{h},\mathfrak{b})=(\mathrm{H},\mathrm{B})$.

Thanks to Lemma 2.29, we have continuous embedding,

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)\hookrightarrow(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}. \tag{2.40}$$

Let us prove the reverse embedding,

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)\hookleftarrow(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}.$$

Without loss of generality, we can assume $s_1 \geqslant \frac{n}{p}$. Let $f \in \mathcal{S}_0(\overline{\mathbb{R}^n_+}) \subset \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, if follows that $f \in (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} \subset \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$. Thus, for all $(a,b) \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$ such that $f = a + b$, we have,

$$b = f - a \in (\mathcal{S}_0(\overline{\mathbb{R}^n_+}) + \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+).$$

In particular, we have $a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)$ and $b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$. Hence, we can introduce $F := \mathrm{E}a + \mathrm{E}b$, where $F_{|_{\mathbb{R}^n_+}} = f$, $\mathrm{E}a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $\mathrm{E}b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, with estimates, given by Corollary 2.19,

$$\|\mathrm{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} \lesssim_{s_0,m,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)} \text{ and } \|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_1,m,p,n} \|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

Then, one may bound the K-functional of F, for $t > 0$,

$$K(t, F, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \leqslant \|\mathrm{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} + t\|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_j,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)} + t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}$$

Taking the infimum over all such functions a and b, we obtain

$$K(t, F, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \lesssim_{s_j,p,n} K(t, f, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)),$$

from which we obtain, after multiplying by $t^{-\theta}$, taking the L^q_*-norm with respect to t, and applying Proposition 2.10,

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \leqslant \|F\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}}.$$

Finally, thanks to the first embedding (2.40), we have

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \sim_{p,s,n} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}}, \forall f \in \mathcal{S}_0(\overline{\mathbb{R}^n_+}).$$

Since $q < +\infty$, we can conclude by density of $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ in both $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ and in the interpolation space $(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}$. Density argument for the later one is carried over by Lemma 2.9 and [BL76, Theorem 3.4.2].

Step 2.2: $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}$, provided $-1 + \frac{1}{p} < s < \frac{1}{p}$, $p \in (1, +\infty)$, $q \in [1, +\infty)$.

Thanks to **Step 1** one may find, $-1 + \frac{1}{p} < s_0 < s < s_1 < \frac{1}{p}$, $\theta \in (0,1)$, such that, as a consequence of [BL76, Theorem 3.4.2], we have the following dense embedding,

$$\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) = \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

The equality in above line is a direct consequence of Proposition 2.15. In this case, the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is a straightforward application of Corollary 2.28 by successive approximations.

Step 2.3: $(\mathfrak{h}, \mathfrak{b}) = (\mathrm{H}_0, \mathrm{B}_{\cdot,\cdot,0})$.

Thanks to Lemma 2.29, we have continuous embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \hookrightarrow \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

We are going to prove the reverse embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \hookleftarrow \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

Again, without loss of generality we can assume $s_1 \geqslant \frac{n}{p}$, otherwise one can go back to **Step 1**. Let us consider $u \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$, then u belongs to $\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. In particular for $(a,b) \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \times \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, such that $u = a + b$ we have

$$b = u - a \in (\mathrm{C}^\infty_c(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n).$$

in particular we have $a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. Consequently, we have $u = \mathcal{P}_0 u = \mathcal{P}_0 a + \mathcal{P}_0 b$, with $\mathcal{P}_0 a \in \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)$ and $\mathcal{P}_0 b \in \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)$, with estimates

$$\|\mathcal{P}_0 a\|_{\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)} \lesssim_{s_0,m,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} \text{ and } \|\mathcal{P}_0 b\|_{\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)} \lesssim_{s_1,m,p,n} \|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)},$$

thanks to Corollary 2.22. Thus, one may follow the lines of **Step 2.1**, to obtain for all $u \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$,

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)} \sim_{s,p,n} \|u\|_{(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q}}.$$

Again, one can conclude via density arguments since $q < +\infty$, and $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$

thanks to **Step 2.2** and Lemma 2.32. ■

The **Step 2.2** in above proof can be turned more formally into,

Corollary 2.34 *Let* $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(-1+\frac{1}{p},\frac{1}{p})$. *Then the following equality holds with equivalence of norms,*

$$\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+).$$

Moreover, the space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ *is dense whenever* $q<+\infty$.

From general interpolation theory we are able to deduce the following,

Corollary 2.35 *Let* $p\in(1,+\infty)$, $s>-1+1/p$, *such that* $(\mathcal{C}_{s,p,\infty})$ *is satisfied.*

- *The space* $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ *is weak* dense in* $\dot{\mathrm{B}}^{s}_{p,\infty,0}(\mathbb{R}^n_+)$.
- *The space* $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ *is weak* dense in* $\dot{\mathrm{B}}^{s}_{p,\infty}(\mathbb{R}^n_+)$.

Proof. — The [BL76, Theorem 3.7.1] with the remark at the end of its proof in combination with Lemma 2.28, with the use of [BL76, Theorem 3.4.2], and Proposition 2.33 imply that, for some $-1+1/p<s_0<s<s_1$, with $\theta\in(0,1)$, such that $s=(1-\theta)s_0+\theta s_1$, we have the following strongly dense embedding,

$$\mathrm{C}^\infty_c(\mathbb{R}^n_+)\hookrightarrow\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)\hookrightarrow(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_\theta,$$

and the following weak* dense embedding

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_\theta\hookrightarrow(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))''_\theta=(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,\infty}=\dot{\mathrm{B}}^{s}_{p,\infty,0}(\mathbb{R}^n_+),$$

so that the result follows. We mention that $(\cdot,\cdot)_\theta$ is the real interpolation functor asking the K-functional to decay at infinity and near the origin, see for instance [Lun18, Definition 1.2].

The same argument apply for the weak* density of $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ in $\dot{\mathrm{B}}^{s}_{p,\infty}(\mathbb{R}^n_+)$. ■

We state below the Besov analogue of Corollary 2.22, Lemma 2.9 and Proposition 2.18, for which the proofs are similar and left to the reader.

Lemma 2.36 *Let* $p_j\in(1,+\infty)$, $q_j\in[1,+\infty]$, $s_j>-1+\frac{1}{p_j}$, $j\in\{0,1\}$, $m\in\mathbb{N}$, *such that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied and* $s_j<m+1+\frac{1}{p_j}$, *and consider the extension operator* E *given by Proposition 2.16.*

Then for all $u\in\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)$, *we have* $\mathrm{E}u\in\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)$, $j\in\{0,1\}$, *with estimate*

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)}\lesssim_{s_j,m,p,n}\|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)}.$$

The same result holds replacing $(\mathrm{E},\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+),\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n))$ *by* $(\mathcal{P}_0,\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n),\dot{\mathrm{B}}^{s_j}_{p_j,q_j,0}(\mathbb{R}^n_+))$, *where* $\mathcal{P}_0$ *is the projection operator given in Lemma 2.21.*

Proposition 2.37 *Let* $p_j\in(1,+\infty)$, $q_j\in[1,+\infty]$, $j\in\{0,1\}$, $-1+\frac{1}{p}<s_0<s_1$, *such that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied. Then the following equality of vector spaces holds with equivalence of norms*

$$\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)=[\dot{\mathrm{B}}^{s_0}_{p_0,q_0}\cap\dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+).$$

In particular, $\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)$ *is a Banach space, and it admits* $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ *as a dense subspace whenever* $q_j<+\infty$, $j\in\{0,1\}$.

Similarly, the following equality with equivalence of norms holds for all $s>0$, $q\in[1,+\infty]$,

$$\mathrm{L}^p(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)=\mathrm{B}^{s}_{p,q}(\mathbb{R}^n_+).$$

With direct consequence similar to Corollary 2.20:

Corollary 2.38 *Let* $p_j\in(1,+\infty)$, $q_j\in[1,+\infty]$ $m_j\in[\![1,+\infty[\![$, $s_j>m_j-1+\frac{1}{p_j}$, $j\in\{0,1\}$, *such that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied. For all* $u\in[\dot{\mathrm{B}}^{s_0}_{p_0,q_0}\cap\dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+)$,

$$\|\nabla^{m_j}u\|_{\dot{\mathrm{B}}^{s_j-m_j}_{p_j,q_j}(\mathbb{R}^n_+)}\sim_{s_j,m_j,p_j,n}\|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)}.$$

Above Proposition 2.37 also implies the expected interpolation result for Besov spaces, for which the proof is similar to the one of Proposition 2.33 and left again to the reader.

Proposition 2.39 *Let* $(p_0, p_1, p, q, q_0, q_1) \in (1,+\infty)^3 \times [1,+\infty]^3$, $s_0, s_1 \in \mathbb{R}$, *such that* $s_0 < s_1$, *and let* $\mathfrak{b} \in \{\mathrm{B}, \mathrm{B}_{\cdot,\cdot,0}\}$, *and set*

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

such that the following assertion is satisfied,

- $s_j > -1 + \frac{1}{p_j}$, $j \in \{0,1\}$, *and* $(\mathcal{C}_{s_0,p_0,q_0})$ *is true;*

Then if $p_0 = p_1 = p$, *and* $(\mathcal{C}_{s,p,q})$ *is satisfied, the following equality holds with equivalence of norms*

$$(\dot{\mathfrak{b}}^{s_0}_{p,q_0}(\mathbb{R}^n_+), \dot{\mathfrak{b}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+). \tag{2.41}$$

If $(\mathcal{C}_{s_0,p_0,q_0})$ *and* $(\mathcal{C}_{s_1,p_1,q_1})$ *are true then also is* $(\mathcal{C}_{s,p_\theta,q_\theta})$ *and with equivalence of norms,*

$$[\dot{\mathfrak{b}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+), \dot{\mathfrak{b}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)]_{\theta} = \dot{\mathfrak{b}}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n_+). \tag{2.42}$$

We finish stating a duality result for homogeneous Besov space son the half-space.

Proposition 2.40 *Let* $p \in (1,+\infty)$, $q \in (1,+\infty]$, $s > -1 + \frac{1}{p}$, *if* $(\mathcal{C}_{s,p,q})$ *is satisfied then the following isomorphisms hold*

$$(\dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_+))' = \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) \text{ and } (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))' = \dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+).$$

Proof. — We only prove $(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))' = \dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, the other equality can be shown in a similar way. First let $q < +\infty$, and choose $u \in \dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, it follows that u induce a linear form on $\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$,

$$v \longmapsto \langle u, \tilde{v}\rangle_{\mathbb{R}^n}$$

where $\tilde{v} \in \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ is any extension of $v \in \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$. If one choose v' to be any other extension of v, we have that $\tilde{v} - v' \in \dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_-)$. Since $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, see either Lemma 2.32 or Corollary 2.34, for $(u_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\mathbb{R}^n_+)$ converging to u, we have

$$\langle u, \tilde{v} - v'\rangle_{\mathbb{R}^n} = \lim_{k\to+\infty} \langle u_k, \tilde{v} - v'\rangle_{\mathbb{R}^n} = 0$$

due to the fact that $\mathbb{R}^n_+ \cap \mathbb{R}^n_- = \emptyset$. Thus, the map does not depend on the choice of the extension but is entirely and uniquely determined by u. We have the continuous canonical embedding

$$\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'.$$

In fact, the same result holds for $q = +\infty$: the space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is sequentially weak* dense in $\dot{\mathrm{B}}^{s}_{p,\infty,0}(\mathbb{R}^n_+)$ by Corollary 2.35.

For the reverse embedding, if $U \in (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'$, it induces a continuous linear functional on $\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ by the mean of

$$v \longmapsto \langle U, \mathbb{1}_{\mathbb{R}^n_+}\tilde{v}\rangle\ ,$$

where again $\tilde{v} \in \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ is any extension of $v \in \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$. Thus, $\mathbb{1}_{\mathbb{R}^n_+}U \in (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n))'$ and by Proposition 2.12 there exists a unique $u \in \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)$ such that, for all $\tilde{v} \in \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$,

$$\langle U, \mathbb{1}_{\mathbb{R}^n_+}\tilde{v}\rangle = \langle u, \tilde{v}\rangle_{\mathbb{R}^n}.$$

Finally, if we test with $\tilde{v} \in \mathrm{C}^\infty_c(\mathbb{R}^n_-)$, it shows that $\operatorname{supp} u \subset \overline{\mathbb{R}^n_+}$, then $u \in \dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$ which close the proof. ■

2.3 Additional notations and some remarks

2.3.1 Operators on Sobolev and Besov spaces

We introduce domains for an operator A acting on Sobolev or Besov spaces, denoting

- $\mathrm{D}^s_p(A)$ (resp. $\dot{\mathrm{D}}^s_p(A)$) its domain on $\mathrm{H}^{s,p}$ (resp. $\dot{\mathrm{H}}^{s,p}$);
- $\mathrm{D}^s_{p,q}(A)$ (resp. $\dot{\mathrm{D}}^s_{p,q}(A)$) its domain on $\mathrm{B}^s_{p,q}$ (resp. $\dot{\mathrm{B}}^s_{p,q}$);
- $\mathrm{D}_p(A) = \mathrm{D}^0_p(A) = \dot{\mathrm{D}}^0_p(A)$ its domain on L^p.

Similarly, $\mathrm{N}^s_p(A)$, $\mathrm{N}^s_{p,q}(A)$ will stand for its nullspace on $\mathrm{H}^{s,p}$ and $\mathrm{B}^s_{p,q}$, and range spaces will be given respectively by $\mathrm{R}^s_p(A)$ and $\mathrm{R}^s_{p,q}(A)$. We replace N and R by $\dot{\mathrm{N}}$ and $\dot{\mathrm{R}}$ for their corresponding corresponding sets on homogeneous function spaces.

If the operator A has different realizations depending on various function spaces and on the considered open set, we may write its domain $\mathrm{D}(A,\Omega)$, and similarly for its nullspace N and range space R. We omit the open set Ω if there is no possible confusion.

2.3.2 Non-exhaustivity of the construction

The goal of presenting here a definitive construction of homogeneous Sobolev and Besov spaces on the half-space is certainly not reached:

- The way arguments are done Section 2.2 always requires a ground function space to intersect with so that it ensures we deal with restriction of elements of $\mathcal{S}'_h(\mathbb{R}^n)$, e.g. see the proof of Proposition 2.18. Hence, with this kind of methods, obtaining more general results like an exhaustive description of dual spaces of homogeneous Besov and Sobolev spaces on $\mathbb{R}^n_+$ in the non-complete case seems to be difficult to reach.
- A related problem is that the used extension operator is not general enough and disallow to recover too much negative index of regularity in case of homogeneous function spaces. It would be of interest to know if one can also recover non-complete positive index independently, without using intersection or density tricks. As mentioned at the beginning of this section, to know if one can construct an operator similar to Rychkov's extension operator, from [Ryc99], $\mathcal{E}$ such that $\mathcal{E}(\mathcal{S}'_h(\overline{\mathbb{R}^n_+})) \subset \mathcal{S}'_h(\mathbb{R}^n)$ with homogeneous estimates would be a sufficiently powerful result to overcome such troubles.
- Other definitions are possible for $\mathcal{S}'_h(\mathbb{R}^n)$. We have chosen here the one with the strongest convergence of low frequencies to continue the work started in [BCD11, Chapter 2] and [DHMT21, Chapter 3]. The choice of possible definitions and their functional analytic consequences on Besov spaces' construction are reviewed by Cobb in [Cob21, Appendix] and [Cob22].

Not to further burden the actual presentation, we just mention that one could also investigate spaces

$$\dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n_+) \text{ and } \dot{\mathcal{B}}^s_{p,\infty,0}(\mathbb{R}^n_+).$$

For those spaces, we have that $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ is dense in the first one by construction, and we can show that $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in the second one, and both may be recovered from interpolation of other appropriate homogeneous Sobolev and Besov spaces. We can also prove corresponding duality and traces results. Details are left to the interested reader.

3 On traces of functions

Dealing with function spaces on domains implies that one may need to investigate the meaning of traces on the boundary if those exist, *i.e.* to see in our setting if the trace operator

$$\gamma_0 \,:\, u \longmapsto u_{|_{\partial\mathbb{R}^n_+}}$$

still has the expected behavior on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$. In fact, in the complete case, it behaves as in the case of inhomogeneous function spaces.

The idea here is to give some appropriate trace theorems for homogeneous Sobolev and Besov spaces since it seems there is no trace theorem for homogeneous function spaces in the literature, except maybe [Jaw78], but in this case corresponding results were obtained in a different framework.

3.1 On inhomogeneous function spaces.

We discuss first about the usual well known trace theorem on $\mathbb{R}^n$ with trace on $\mathbb{R}^{n-1}\times\{0\}$ in the inhomogeneous case, the result is a rewritten weaker version adapted to our context.

Theorem 3.1 ([BL76, Theorem 6.6.1]) *Let $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(\frac{1}{p},+\infty)$, and consider the following operator*

$$\gamma_0 \,:\, \left\{ \begin{array}{ll} \mathcal{S}(\mathbb{R}^n) & \longrightarrow \mathcal{S}(\mathbb{R}^{n-1}) \\ \quad u & \longmapsto \;\; u(\cdot,0) \end{array} \right. ,$$

then following statements are true:

(i) the trace operator $\gamma_0 : \mathrm{H}^{s,p}(\mathbb{R}^n) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ is a bounded surjection, in particular for all $u\in \mathrm{H}^{s,p}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)};$$

(ii) the trace operator $\gamma_0 : \mathrm{B}^{s}_{p,q}(\mathbb{R}^n) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ is a bounded surjection, in particular for all $u\in \mathrm{B}^{s}_{p,q}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s,p,n,q} \|u\|_{\mathrm{B}^{s}_{p,q}(\mathbb{R}^n)};$$

(iii) the trace operator $\gamma_0 : \mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n) \longrightarrow \mathrm{L}^p(\mathbb{R}^{n-1})$ is a bounded surjection, in particular for all $u\in \mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{p,n} \|u\|_{\mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)};$$

Moreover the trace operator γ_0 admits a linear right bounded inverse Ext *in cases (i) and (ii).*

Remark 3.2 One also mention [Sch10, Theorems 2.2 & 2.10], [JW84, Sections V-VII], which give different proofs of the trace theorem. Notice that in [Sch10, Theorems 2.2 & 2.10] and [Saw18, Theorems 4.47, 4.48] the right bounded inverse they give is not linear but covers case (iii).

Proof. — We only give the proof for the right bounded inverse. The idea here is to complete the approach given in [BL76, Exercices 25, 26, p.166] to recover the full range of exponents since $(\frac{1}{p},1]$ were missing. To do so let $p\in(1,+\infty)$, $s>\frac{1}{p}$, and $m\in\mathbb{N}$ such that $s<m+1+\frac{1}{p}$. Consider $\chi\in \mathrm{C}^\infty_c(\mathbb{R})$ such that $\operatorname{supp}\chi\subset[-1,1]$, $0\leqslant\chi\leqslant 1$ and $\chi(0)=1$.

We introduce the following operator,

$$\mathcal{L}_+ \,:\, f\longmapsto \left[(x',x_n)\mapsto \chi(x_n)e^{-x_n(-\Delta')^{\frac{1}{2}}}f(x')\right].$$

Since $\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) = \mathrm{L}^p(\mathbb{R}^{n-1}) + \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, see [BL76, Theorem 2.7.1, Theorem 6.3.2], we can apply Lemma B.1, so that for all $f=a+b\in \mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, where $(a,b)\in \mathrm{L}^p(\mathbb{R}^{n-1})\times \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\begin{aligned} \|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} &= \left(\int_0^{+\infty} \|\chi(x_n)e^{-x_n(-\Delta')^{\frac{1}{2}}}f\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})}\,\mathrm{d}x_n\right)^{\frac{1}{p}} \\ &= \left(\int_0^{+\infty} \left(t^{\frac{1}{p}}\|\chi(t)e^{-t(-\Delta')^{\frac{1}{2}}}f\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\ &\leqslant \left(\int_0^{1} \|e^{-t(-\Delta')^{\frac{1}{2}}}a\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})}\mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_0^{+\infty} \left(t^{\frac{1}{p}}\|e^{-t(-\Delta')^{\frac{1}{2}}}b\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\ &\lesssim_{p,n} \|a\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} + \|b\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}, \end{aligned}$$

thus, one may take the infimum on all such pair (a,b) to obtain,

$$\|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

Now, we can use the higher order reflection extension operator E introduced in the proof of Proposition 2.16 to define $\mathcal{L} := \mathrm{E}\mathcal{L}_+$. Thus, due to above boundedness properties, it follows that

$$\|\mathcal{L} f\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n,m} \|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

It has been proved, see [BL76, Exercises 25, 26, p.166], that $\mathcal{L}$ also sastifies, for all $f \in \mathrm{B}^{k-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|\mathcal{L} f\|_{\mathrm{H}^{k,p}(\mathbb{R}^n)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{k-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})},$$

for all $1 \leqslant k \leqslant m+1$. Finally, the result follows by complex and real interpolation, and $\mathrm{Ext} = \mathcal{L}$ is the desired right bounded inverse. ■

Remark 3.3 In the above proof, the extension operator from the boundary to the whole space depends on some fixed regularity degree, which make it non-universal. If one wants an universal extension operator from the boundary to the whole space, one may replace the use of E from the proof of Proposition 2.16 by the use of Stein's extension operator on the half-space, check [Ste70, Section VI, Theorem 5'].

Corollary 3.4 *Let* $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in (\frac{1}{p},+\infty)$, *we have continuous embeddings:*

(i) $\mathrm{H}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$;

(ii) $\mathrm{B}^{s}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$;

(iii) $\mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1}))$;

(iv) $\mathrm{B}^{s}_{p,\infty}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{b,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,\infty}(\mathbb{R}^{n-1}) - \mathit{weak}^*)$.

Proof. — We only check validity of the embedding

$$\mathrm{H}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

Let $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, for $t>0$, for almost every $x=(x',x_n) \in \mathbb{R}^n$, we introduce $u_t(x',x_n) := u(x',x_n+t)$, we have $u_t \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, and by Theorem 3.1,

$$\|\gamma_0 u_t\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)},$$

$$\|\gamma_0 (u_t - u)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u_t - u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}.$$

Therefore, by strong continuity of translation in Lebesgue spaces, then in Sobolev spaces, we obtain

$$\|\gamma_0 (u_t - u)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u_t - u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} \xrightarrow[t\to 0]{} 0.$$

Hence, $u \in \mathrm{C}^0_{b,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$, with estimate,

$$\left\| t \mapsto \|u(\cdot,t)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \right\|_{\mathrm{L}^\infty(\mathbb{R})} \lesssim_{p,s,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}.$$

Finally, one can approximate u by Schwartz functions to deduce

$$u \in \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

One may perform a similar proof for all other cases, and one may check [Gui91, Proposition 1.9] for the continuity of translation in Besov spaces, one may also use a density and an interpolation argument. ■

3.2 On homogeneous function spaces.

Theorem 3.5 *Let* $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (\frac{1}{p},+\infty)$, *then for* $(\mathfrak{h},\mathfrak{b}) \in \{(\mathrm{H},\mathrm{B}), (\dot{\mathrm{H}},\dot{\mathrm{B}})\}$, *we consider the trace operator*

$$\gamma_0 \,:\, u \longmapsto u(\cdot,0).$$

The following assertions are true.

(i) *For all* $u \in \mathfrak{h}^{s,p}(\mathbb{R}^n_+)$, *we have* $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \mathfrak{b}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$, *with estimate*

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathfrak{b}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{h}^{s,p}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends boundedly as $\gamma_0 \,:\, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \to \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ *whenever* $(\mathcal{C}_{s,p})$ *is satisfied, and the following continuous embedding holds*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

(ii) *For all* $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, *we have* $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \mathfrak{b}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$, *with estimate*

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathfrak{b}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{b}^{s}_{p,q}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends boundedly as $\gamma_0 \,:\, \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) \to \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ *whenever* $(\mathcal{C}_{s,p,q})$ *is satisfied, and the following continuous embedding holds*

$$\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})).$$

If $q = +\infty$, *the result still holds with uniform boundedness and weak* continuity only.*

(iii) *For all* $u \in \mathrm{B}^{1/p}_{p,1}(\mathbb{R}^n_+)$, *we have* $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \mathrm{L}^p(\mathbb{R}^{n-1}))$, *with estimate*

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathrm{L}^p(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{b}^{1/p}_{p,1}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends boundedly as $\gamma_0 \,:\, \dot{\mathrm{B}}^{1/p}_{p,1}(\mathbb{R}^n_+) \to \mathrm{L}^p(\mathbb{R}^{n-1})$ *and the following continuous embedding holds*

$$\dot{\mathrm{B}}^{1/p}_{p,1}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \mathrm{L}^p(\mathbb{R}^{n-1})).$$

Moreover,

(a) *If* $(\mathfrak{h},\mathfrak{b}) = (\mathrm{H},\mathrm{B})$, *the trace operator* γ_0 *admits a linear right bounded inverse* $\mathrm{Ext}_{\mathbb{R}^n_+}$ *in cases (i) and (ii).*

(b) *If* $(\mathfrak{h},\mathfrak{b}) = (\dot{\mathrm{H}},\dot{\mathrm{B}})$, *the trace operator* γ_0 *admits a linear right bounded inverse* $\underline{\mathrm{Ext}}_{\mathbb{R}^n_+}$ *in cases (i) and (ii).*

Proof. — We cut the proof in several steps.

Step 1: The case $(\mathfrak{h},\mathfrak{b}) = (\mathrm{H},\mathrm{B})$.

The result is a direct consequence of Corollary 3.1, and the definition of functions space by restriction.

One choose $\mathrm{Ext}_{\mathbb{R}^n_+} = \mathcal{L}_+$ introduced in the proof of Theorem 3.1 which satisfies the desired boundedness properties.

Step 2.1: The case $(\mathfrak{h},\mathfrak{b}) = (\dot{\mathrm{H}},\dot{\mathrm{B}})$. Boundedness of the trace operator.

We only achieve the case *(ii)* other ones can be done similarly. From **Step 1**, and for fixed $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s > \frac{1}{p}$, and $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, we have

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)}.$$

Thus, one may use the fact that $\mathrm{B}^s_{p,q}(\mathbb{R}^n_+) = \mathrm{L}^p(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, which comes from Proposition 2.37, to obtain

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

So that, by a dilation argument, replacing u, by $u_\lambda := u(\lambda\cdot)$, for $\lambda \in 2^{\mathbb{N}}$,

$$\lambda^{s-\frac{n}{p}}\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n,q} \lambda^{-\frac{n}{p}}\|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + \lambda^{s-\frac{n}{p}}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

Hence, we can divide by $\lambda^{s-\frac{n}{p}}$ on both sides and pass to the limit $\lambda \longrightarrow +\infty$,

$$\lambda^{s-\frac{n}{p}}\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n,q} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

Therefore, if $q < +\infty$, and $(\mathcal{C}_{s,p,q})$ is satisfied, the embedding

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$$

holds by density. If $q = +\infty$, and $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows from real interpolation.

Step 2.2: The case $(\mathfrak{h},\mathfrak{b}) = (\dot{\mathrm{H}},\dot{\mathrm{B}})$. Boundedness of the extension operator.

The operator T given by Proposition B.2 is an appropriate extension operator which satisfies the desired boundedness properties. Thus $\underline{\mathrm{Ext}}_{\mathbb{R}^n_+} := T$ behaves as expected. ■

A raised question is about what happens when we want to deal with intersection of homogeneous Sobolev and Besov spaces.

Proposition 3.6 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $-1+\frac{1}{p} < s_0 < \frac{1}{p} < s_1$, and $\theta \in (0,1)$ such that*

$$\frac{1}{p} = (1-\theta)s_0 + \theta s_1.$$

Then,

(i) For all $u \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$, we have $\gamma_0 u \in \mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, with estimate

$$\|\gamma_0 u\|_{\mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)}\|u\|^{\theta}_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

We also have,

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}\ ;$$

(ii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)$, we have $\gamma_0 u \in \mathrm{B}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$, with estimate

$$\|\gamma_0 u\|_{\mathrm{B}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)}\|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)}.$$

We also have,

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)};$$

(iii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,\infty}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p,\infty}(\mathbb{R}^n_+)$, we have $\gamma_0 u \in \mathrm{L}^p(\mathbb{R}^{n-1})$, with estimate

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,\infty}(\mathbb{R}^n_+)}\|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,\infty}(\mathbb{R}^n_+)}.$$

Proof. — We only start proving the point *(ii)*, and claim that point *(i)* can be achieved in a similar manner. We start noticing, the following continuous embedding,

$$\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+) \overset{\iota}{\hookrightarrow} (\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+),\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+))_{\theta,1} = \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n_+) \overset{\gamma_0}{\hookrightarrow} \mathrm{L}^p(\mathbb{R}^{n-1}).$$

Here, ι is the canonical embedding obtained via standard interpolation theory, and the last embedding via the trace operator is a direct consequence of Theorem 3.5, and everything can be turned into the following inequality,

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)}\|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)},\ \forall u \in \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+).$$

Again, from Theorem 3.5 we obtain for all $u \in \mathcal{S}_0(\overline{\mathbb{R}^n_+})$,

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_1,p,n} \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)}.$$

Then one may sum both inequality, notice that $\mathrm{L}^p(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) = \mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ and use the density argument provided by Proposition 2.37 so that each estimate holds. ■

Remark 3.7 As in Theorem 3.5, above Proposition 3.6 could be turned into a C^0_{0,x_n}-embedding in the appropriate Besov space.

Proposition 3.8 *Let $p_j \in (1,+\infty)$, $q_j \in [1,+\infty)$, $s_j > 1/p_j$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ (resp. $(\mathcal{C}_{s_0,p_0,q_0})$) is satisfied. Then,*

(i) For all $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$, we have $\gamma_0 u \in \dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})$, $j \in \{0,1\}$, with estimate

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})} \lesssim_{s_j,p_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)};$$

(ii) For all $u \in [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+)$, we have $\gamma_0 u \in \dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})$, $j \in \{0,1\}$, with estimate

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})} \lesssim_{s_j,p_j,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)};$$

Remark 3.9 Corollary B.3 yields the ontoness of the trace operator on intersection spaces given by above Proposition 3.8.

Lemma 3.10 *Let $p_j \in (1,+\infty)$, $s \in (1/p_j, 1+1/p_j)$, $j \in \{0,1\}$ such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. For all $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+,\mathbb{C})$ such that $u_{|_{\partial\mathbb{R}^n_+}} = 0$, the extension $\tilde{u}$ to $\mathbb{R}^n$ by 0, satisfies*

$$\tilde{u} \in [\dot{\mathrm{H}}^{s_0,p_0}_0 \cap \dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C})$$

with estimate

$$\|\tilde{u}\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{p_j,s_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)},\ j \in \{0,1\}.$$

The result still holds replacing $\dot{\mathrm{H}}^{s_j,p_j}$ by $\dot{\mathrm{B}}^{s_j}_{p_j,q_j}$, $q_j \in [1,+\infty]$, $j \in \{0,1\}$ assuming that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied.

Proof. — Let $u \in [\dot{\mathrm{H}}^{s_0,p_0}_0 \cap \dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C})$ such that $u_{|_{\partial\mathbb{R}^n_+}} = 0$, then for all $\phi \in [\dot{\mathrm{H}}^{1-s_j,p_j'} \cap \mathcal{S}](\mathbb{R}^n,\mathbb{C}^n)$, we have

$$\int_{\mathbb{R}^n_+} \nabla u \cdot \phi = -\int_{\mathbb{R}^n_+} \mathrm{div}\,(\phi)\, u.$$

So that introducing the extensions by 0 to $\mathbb{R}^n$, $\tilde{u}$ and $\widetilde{\nabla u}$,

$$\int_{\mathbb{R}^n} \widetilde{\nabla u} \cdot \phi = \int_{\mathbb{R}^n_+} \nabla u \cdot \phi = -\int_{\mathbb{R}^n_+} \mathrm{div}\,(\phi)\, u = -\int_{\mathbb{R}^n} \mathrm{div}\,(\phi)\, \tilde{u} = \big\langle \nabla \tilde{u}, \phi \big\rangle_{\mathbb{R}^n}.$$

Therefore, for all $\phi \in [\dot{\mathrm{H}}^{1-s_j,p_j'} \cap \mathcal{S}](\mathbb{R}^n,\mathbb{C}^n)$,

$$\int_{\mathbb{R}^n} \widetilde{\nabla u} \cdot \phi = \big\langle \nabla \tilde{u}, \phi \big\rangle_{\mathbb{R}^n}.$$

Hence $\widetilde{\nabla u} = \nabla \tilde{u}$ in $\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^n)$. Thus, by Propositions 2.11 and 2.15, we deduce that

$$\begin{aligned}\big|\big\langle \nabla \tilde{u}, \phi \big\rangle_{\mathbb{R}^n}\big| &\leqslant \|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)} \|\widetilde{\nabla u}\|_{\dot{\mathrm{H}}^{s_j-1,p_j}(\mathbb{R}^n)} \\ &\lesssim_{p_j,n,s_j} \|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)} \|\nabla u\|_{\dot{\mathrm{H}}^{s_j-1,p_j}(\mathbb{R}^n_+)} \\ &\lesssim_{p_j,n,s_j} \|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.\end{aligned}$$

One may conclude thanks to Proposition 2.11, and Corollary 2.20. The case of Besov spaces follows the same lines. ■

The following corollary is then immediate

Corollary 3.11 *Let* $p_j \in (1,+\infty)$, $s_j \in (1/p_j, 1+1/p_j)$, $j \in \{0,1\}$ *such that* $(\mathcal{C}_{s_0,p_0})$ *is satisfied. We have the following canonical isomorphism of Banach spaces*

$$\{\, u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+,\mathbb{C}) \,|\, u_{|_{\partial\mathbb{R}^n_+}} = 0 \,\} \simeq [\dot{\mathrm{H}}^{s_0,p_0}_0 \cap \dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C}).$$

The result still holds replacing $\dot{\mathrm{H}}^{s_j,p_j}$ *by* $\dot{\mathrm{B}}^{s_j}_{p_j,q_j}$, $q_j \in [1,+\infty]$, $j \in \{0,1\}$ *assuming that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied.*

4 Applications: the Dirichlet and Neumann Laplacians on the half-space

We introduce the following subsets of the complex plane

$$\Sigma_\mu := \{\, z \in \mathbb{C}^* \,:\, |\arg(z)| < \mu \,\}, \text{ if } \mu \in (0,\pi),$$

we also define $\Sigma_0 := (0,+\infty)$, and later we are going to consider $\overline{\Sigma}_\mu$ its closure.

An operator $(\mathrm{D}(A), A)$ on complex valued Banach space X is said to be ω-***sectorial***, if for a fixed $\omega \in (0,\pi)$, both conditions are satisfied

(i) $\sigma(A) \subset \overline{\Sigma}_\omega$, where $\sigma(A)$ stands for the spectrum of A ;

(ii) For all $\mu \in (\omega,\pi)$, $\sup_{\lambda\in\mathbb{C}\setminus\overline{\Sigma}_\mu} \|\lambda(\lambda\mathrm{I} - A)^{-1}\|_{X\to X} < +\infty$.

Sectorial operators is widely reviewed in several references but we mention here Haase's book [Haa06]. One may also check [Ege15, Chapter 3].

Before starting the analysis of Dirichlet, Neumannon the half space, we introduce two appropriate extension operators. We denote $\mathrm{E}_\mathcal{J}$, for $\mathcal{J} \in \{\mathcal{D},\mathcal{N}\}$, the extension operator defined for any measurable function u on $\mathbb{R}^n_+$, for almost every $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$:

$$\mathrm{E}_\mathcal{D} u(x',x_n) := \begin{cases} u(x',x_n) & , \text{ if } (x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}_+, \\ -u(x',-x_n) & , \text{ if } (x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}^*_- \ ; \end{cases}$$

$$\mathrm{E}_\mathcal{N} u(x',x_n) := \begin{cases} u(x',x_n) & , \text{ if } (x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}_+, \\ u(x',-x_n) & , \text{ if } (x',x_n) \in \mathbb{R}^{n-1}\times\mathbb{R}^*_-. \end{cases}$$

Obviously, for $\mathcal{J} \in \{\mathcal{D},\mathcal{N}\}$, $s \in (-1+1/p, 1/p)$, $p \in (1,+\infty)$, the Proposition 2.15 leads to boundedness of

$$\mathrm{E}_\mathcal{J} \,:\, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n). \tag{4.1}$$

The same result holds replacing $\dot{\mathrm{H}}^{s,p}$ by either $\mathrm{H}^{s,p}$, $\mathrm{B}^s_{p,q}$, or even by $\dot{\mathrm{B}}^s_{p,q}$, $q \in [1,+\infty]$.

We are going to use the properties of Laplacian acting on the whole space to build resolvent estimates for both the the Dirichlet and the Neumann Laplacian. Usual Dirichlet and Neumann Laplacians are the operators $(\mathrm{D}(\Delta_\mathcal{J}), -\Delta_\mathcal{J})$, for $\mathcal{J} \in \{\mathcal{D},\mathcal{N}\}$, where the subscript $\mathcal{D}$ (resp. $\mathcal{N}$) stands for the Dirichlet (resp. Neumann) Laplacian, with, for $p \in (1,+\infty)$,

$$\mathrm{D}_p(\Delta_\mathcal{D}) := \left\{ u \in \mathrm{H}^{1,p}(\mathbb{R}^n_+,\mathbb{C}) \,\middle|\, \Delta u \in \mathrm{L}^p(\mathbb{R}^n_+,\mathbb{C}) \text{ and } u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\},$$

$$\mathrm{D}_p(\Delta_\mathcal{N}) := \left\{ u \in \mathrm{H}^{1,p}(\mathbb{R}^n_+,\mathbb{C}) \,\middle|\, \Delta u \in \mathrm{L}^p(\mathbb{R}^n_+,\mathbb{C}) \text{ and } \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\}.$$

For $\mathcal{J} \in \{\mathcal{D},\mathcal{N}\}$, and all $u \in \mathrm{D}_p(\Delta_\mathcal{J})$,

$$-\Delta_\mathcal{J} u := -\Delta u.$$

When $p = 2$, one can also realize both Dirichlet and Neumann Laplacians by the mean of densely defined, symmetric, accretive, continuous, closed, sesquilinear forms on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, for $\mathcal{J} \in \{\mathcal{D},\mathcal{N}\}$,

$$\mathfrak{a}_\mathcal{J} \,:\, \mathrm{D}_2(\mathfrak{a}_\mathcal{J})^2 \ni (u,v) \longmapsto \int_{\mathbb{R}^n_+} \nabla u \cdot \overline{\nabla v} \tag{4.2}$$

with $\mathrm{D}_2(\mathfrak{a}_{\mathcal{D}}) = \mathrm{H}^1_0(\mathbb{R}^n_+,\mathbb{C})$, $\mathrm{D}_2(\mathfrak{a}_{\mathcal{N}}) = \mathrm{H}^1(\mathbb{R}^n_+,\mathbb{C})$, so that it is easy to see, and well-known, that both, the Neumann and Dirichlet Laplacians, are closed, densely defined, non-negative self-adjoint operators on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, see [Ouh05, Chapter 1, Section 1.2]. We can be even more precise.

Proposition 4.1 *Provided $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, the operator $(\mathrm{D}_2(\Delta_{\mathcal{J}}),-\Delta_{\mathcal{J}})$ is an injective non-negative self-adjoint and 0-sectorial operator on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$.*

Moreover, the following hold

(i) $\mathrm{D}_2(\Delta_{\mathcal{J}})$ is a closed subspace of $\mathrm{H}^2(\mathbb{R}^n_+,\mathbb{C})$;

(ii) Provided $\mu\in[0,\pi)$, for $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, then $u := (\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f$ satisfies

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} \lesssim_{n,\mu} \|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)} \ ;$$

(iii) The following resolvent identity holds for all $\mu\in[0,\pi)$, $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$,

$$\mathrm{E}_{\mathcal{J}}(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f = (\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{J}}f.$$

Remark 4.2 For $u:\mathbb{R}^n_+\longrightarrow\mathbb{C}$, we set

$$\tilde{u}_{\mathcal{J}} := [\mathrm{E}_{\mathcal{J}}u]_{|_{\mathbb{R}^n_-}}$$

for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$. We notice that in $\mathcal{D}'(\mathbb{R}^n_-,\mathbb{C})$,

$$\partial_{x_n}[\tilde{u}_{\mathcal{N}}] = \widetilde{[\partial_{x_n}u]}_{\mathcal{D}} \text{ and } \partial_{x_n}[\tilde{u}_{\mathcal{D}}] = \widetilde{[\partial_{x_n}u]}_{\mathcal{N}}.$$

Proof. — One may use self-adjointness and (4.2) which gives, by standard hilbertian theory, the following resolvent estimate

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} \lesssim_{\mu} \|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)},$$

where $u := (\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, $\lambda\in\Sigma_\mu$, $\mu\in[0,\pi)$.

Now, for fixed $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, $\lambda\in\Sigma_\mu$, $\mu\in[0,\pi)$, we consider $u := (\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f$. Assuming $\mathcal{J}=\mathcal{N}$, we have for $\phi\in\mathcal{S}(\mathbb{R}^n,\mathbb{C})$,

$$\begin{aligned}\big\langle \mathrm{E}_{\mathcal{N}}u, -\Delta\phi\big\rangle_{\mathbb{R}^n} &= \big\langle u, -\Delta\phi\big\rangle_{\mathbb{R}^n_+} + \big\langle \tilde{u}_{\mathcal{N}}, -\Delta\phi\big\rangle_{\mathbb{R}^n_-}\\ &= \big\langle \nabla u, \nabla\phi\big\rangle_{\mathbb{R}^n_+} + \big\langle u, \nabla\phi\cdot\mathfrak{e}_{\mathfrak{n}}\big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \tilde{u}_{\mathcal{N}}, \nabla\phi\cdot\mathfrak{e}_{\mathfrak{n}}\big\rangle_{\partial\mathbb{R}^n_-}\\ &\quad + \big\langle \widetilde{[\nabla' u]}_{\mathcal{N}}, \nabla'\phi\big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[\partial_{x_n}u]}_{\mathcal{D}}, \partial_{x_n}\phi\big\rangle_{\mathbb{R}^n_-}\end{aligned}$$

Since $\partial\mathbb{R}^n_+ = \partial\mathbb{R}^n_- = \mathbb{R}^{n-1}\times\{0\}$, with traces $\tilde{u}_{\mathcal{N}|_{\partial\mathbb{R}^n_-}} = u_{|_{\partial\mathbb{R}^n_+}}$, we deduce $\big\langle u_{|_{\partial\mathbb{R}^n_+}}, \nabla\phi\cdot\mathfrak{e}_{\mathfrak{n}}\big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \tilde{u}_{\mathcal{N}|_{\partial\mathbb{R}^n_-}}, \nabla\phi\cdot\mathfrak{e}_{\mathfrak{n}}\big\rangle_{\partial\mathbb{R}^n_-} = 0$. Then, thanks to Remark 4.2 and the boundary condition on u, *i.e.* $\partial_{x_n}u_{|_{\partial\mathbb{R}^n_+}} = 0$, we have

$$\begin{aligned}\big\langle \mathrm{E}_{\mathcal{N}}u, -\Delta\phi\big\rangle_{\mathbb{R}^n} &= \big\langle \nabla u, \nabla\phi\big\rangle_{\mathbb{R}^n_+} + \big\langle \widetilde{[\nabla' u]}_{\mathcal{N}}, \nabla'\phi\big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[\partial_{x_n}u]}_{\mathcal{D}}, \partial_{x_n}\phi\big\rangle_{\mathbb{R}^n_-}\\ &= \big\langle -\Delta u, \phi\big\rangle_{\mathbb{R}^n_+} + \big\langle \widetilde{[-\Delta' u]}_{\mathcal{N}}, \phi\big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[-\partial^2_{x_n}u]}_{\mathcal{N}}, \phi\big\rangle_{\mathbb{R}^n_-}\\ &\quad - \big\langle \partial_{x_n}u, \phi\big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \widetilde{[\partial_{x_n}u]}_{\mathcal{D}}, \phi\big\rangle_{\partial\mathbb{R}^n_-}\\ &= \big\langle \mathrm{E}_{\mathcal{N}}[-\Delta u], \phi\big\rangle_{\mathbb{R}^n}.\end{aligned}$$

Thus, $-\Delta\mathrm{E}_{\mathcal{N}}u = \mathrm{E}_{\mathcal{N}}[-\Delta u]$ in $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$. One may reproduce above calculations for $\mathcal{J}=\mathcal{D}$. So for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, $\mathrm{E}_{\mathcal{J}}u$ is a solution of

$$\lambda U - \Delta U = \mathrm{E}_{\mathcal{J}}f.$$

We have $\mathrm{E}_{\mathcal{J}}f\in\mathrm{L}^2(\mathbb{R}^n,\mathbb{C})$. By uniqueness of the solution provided in $\mathbb{R}^n$, we necessarily have $U=\mathrm{E}_{\mathcal{J}}u$, which be can written as

$$\mathrm{E}_{\mathcal{J}}(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f = (\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{J}}f.$$

Thus one deduces point *(iii)*, from the definition of function spaces by restriction, *(ii)* follows, and finally setting $\lambda = 1$ in point *(ii)* yields *(i)*. ■

We want to show some sharp regularity results on the Dirichlet an Neumann resolvent problems, on the scale of inhomogeneous and homogeneous Sobolev and Besov spaces. To do so, we introduce their corresponding domains on each space. Provided $p \in (1,+\infty)$ $s \in (-1+1/p, 1+1/p)$, if is satisfied $(\mathcal{C}_{s,p})$:

$$\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{D}}) := \left\{ u \in [\dot{\mathrm{H}}^{s,p}_0 \cap \dot{\mathrm{H}}^{s+1,p}](\mathbb{R}^n_+,\mathbb{C}) \,\middle|\, \Delta u \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+,\mathbb{C}) \text{ and } u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\} \subset \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+,\mathbb{C}),$$
$$\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{N}}) := \left\{ u \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+1,p}](\mathbb{R}^n_+,\mathbb{C}) \,\middle|\, \Delta u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}) \text{ and } \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}).$$

We can also consider their domains on inhomogeneous Sobolev and Besov spaces, as well as homogeneous spaces, replacing $(\dot{\mathrm{D}}^s_p, \dot{\mathrm{H}}^{s,p})$ by either $(\mathrm{D}^s_p, \mathrm{H}^{s,p})$, $(\mathrm{D}^s_{p,q}, \mathrm{B}^s_{p,q})$ and finally $(\dot{\mathrm{D}}^s_{p,q}, \dot{\mathrm{B}}^s_{p,q})$ provided $q \in [1,+\infty]$, and $(\mathcal{C}_{s,p,q})$ is satisfied.

It is then not difficult to see that the Dirichlet and Neumann Laplacians are well defined unbounded closed linear operators, densely defined, if $q \in [1,+\infty)$ in the case of inhomogeneous and homogeneous Besov spaces. If $q = +\infty$, the domain of the Dirichlet (resp. Neumann) Laplacian is only known to be weak* dense in $\mathrm{B}^s_{p,\infty,0}$ (resp. in $\mathrm{B}^s_{p,\infty}$) and $\dot{\mathrm{B}}^s_{p,\infty,0}$ (resp. $\dot{\mathrm{B}}^s_{p,\infty}$).

Proposition 4.3 *Let $p, \tilde{p} \in (1,+\infty)$, $q, \tilde{q} \in [1,+\infty]$, $s \in (-1+\frac{1}{p}, 1+\frac{1}{p})$, $s \neq 1/p$, $\alpha \in (-1+\frac{1}{\tilde{p}}, 1+\frac{1}{\tilde{p}})$, $\alpha \neq 1/\tilde{p}$, and $\lambda \in \Sigma_\mu$ provided $\mu \in [0,\pi)$. For $f \in \mathcal{S}'_h(\overline{\mathbb{R}^n_+},\mathbb{C})$, let us consider the* ***resolvent Dirichlet*** *problem with homogeneous boundary condition:*

$$\begin{cases} \lambda u - \Delta u &= f, \quad \text{in } \mathbb{R}^n_+, \\ u_{|_{\partial\mathbb{R}^n_+}} &= 0, \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{$\mathcal{DL}_\lambda$}$$

Provided $(\mathcal{C}_{s,p})$ is satisfied, if $f \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+,\mathbb{C})$, then the problem $(\mathcal{DL}_\lambda)$ admits a unique solution $u \in [\dot{\mathrm{H}}^{s,p}_0 \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+,\mathbb{C})$ with estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

If moreover $\alpha \neq 1/\tilde{p}$ and $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}_0(\mathbb{R}^n_+,\mathbb{C})$, then we also have $u \in [\dot{\mathrm{H}}^{\alpha,\tilde{p}} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n_+,\mathbb{C})$ with corresponding estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha,\mu} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)}.$$

The result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}})$ by $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^\alpha_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}})$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

Remark 4.4 We allowed us a slight abuse of notation here: we identified $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$ with either

- $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ when $s \in (-1+1/p, 1/p)$, thanks to Proposition 2.15;
- $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ with homogeneous Dirichlet boundary condition when $s \in (1/p, 1+1/p)$, thanks to Corollary 3.11.

The same identification is made for Besov spaces, and inhomogeneous function spaces.

Proof. — Provided $p \in (1,+\infty)$, and firstly that $s \in (-1+1/p, 1/p)$, for $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, it follows from Proposition 2.15 that for $U := (\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{D}} f$

$$|\lambda| \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}} \|\nabla U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\nabla^2 U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Thus, by definition of restriction restriction space, we set $u := U_{|_{\mathbb{R}^n_+}}$ which satisfies

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$

then the map $f \mapsto [(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{D}}f]_{|_{\mathbb{R}^n_+}}$ is a bounded map on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$. Everything goes similarly for $\mathrm{H}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$. One may check, as in the proof of Proposition 4.1, and by a limiting argument, given the density of $[\mathrm{L}^2\cap\dot{\mathrm{H}}^{s,p}](\mathbb{R}^n_+,\mathbb{C})$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, that $u_{|_{\partial\mathbb{R}^n_+}}=0$, and

$$\lambda u - \Delta u = f \text{ in } \mathbb{R}^n_+.$$

Again, as in the proof of Proposition 4.1, one may check that any solution u to above resolvent Dirichlet problem necessarily satisfies $\mathrm{E}_{\mathcal{D}}u = (\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{D}}f$.

Now if $s\in(1/p,1+1/p)$, $f\in[\dot{\mathrm{H}}^{s-1,p}_0\cap\dot{\mathrm{H}}^{s,p}_0](\mathbb{R}^n_+,\mathbb{C})$ then we have, thanks to previous considerations, $U:=(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{D}}f\in\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n,\mathbb{C})$. It suffices to show that $U\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\mathbb{C})$, which is true. Indeed, we have

$$\begin{aligned}|\lambda|\|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} &\lesssim_{s,p,n,\mu} |\lambda|\|\nabla U\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)}\\ &\lesssim_{s,p,n,\mu} \|\nabla \mathrm{E}_{\mathcal{D}}f\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)}\\ &\lesssim_{s,p,n,\mu} \sum_{k=1}^{n}\|\partial_{x_k}\mathrm{E}_{\mathcal{D}}f\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)}.\end{aligned}$$

Since equalities $\partial_{x_k}\mathrm{E}_{\mathcal{D}}f=\mathrm{E}_{\mathcal{D}}\partial_{x_k}f$, $k\in[\![1,n-1]\!]$ and $\partial_{x_n}\mathrm{E}_{\mathcal{D}}f=\mathrm{E}_{\mathcal{N}}\partial_{x_n}f$ occur in $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$, we deduce

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \leqslant |\lambda|\|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

One may proceed similarly as before to obtain the full estimate

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Thus the estimates still hold by density for all $f\in\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$, $s\in(-1+1/p,1+1/p)$, $s\neq 1/p$, whenever $(\mathcal{C}_{s,p})$ is satisfied.

The $\dot{\mathrm{H}}^{\alpha,\tilde{p}}$-estimate for $f\in[\dot{\mathrm{H}}^{s,p}_0\cap\dot{\mathrm{H}}^{\alpha,\tilde{p}}_0](\mathbb{R}^n_+)$ can be obtained the same way, whenever $(\mathcal{C}_{s,p})$ is satisfied.

The case of Besov spaces $\dot{\mathrm{B}}^{s}_{p,q,0}$ can be achieved via similar argument for $q<+\infty$, the case $q=+\infty$ is obtained via real interpolation. The case of the $\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q},0}$-estimate for $f\in\dot{\mathrm{B}}^{s}_{p,q,0}\cap\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q},0}$ can be done as above. ■

Remark 4.5 We have excluded cases $s=1/p$ and $\alpha=1/\tilde{p}$. Both require to introduce, e.g. in case of Sobolev spaces, the homogeneous counterpart of the Lions-Magenes Sobolev space $\dot{\mathrm{H}}^{1/q,q}_{00}(\mathbb{R}^n_+)$, $q\in\{p,\tilde{p}\}$. See for instance [LM72, Chapter 1, Theorem 11.7] for the inhomogeneous space in the case $q=2$.

The proof for the Neumann resolvent problem in the proposition below is fairly similar to the proof of Proposition 4.3, a complex interpolation argument allows values $s=1/p$ and $\alpha=1/\tilde{p}$.

Proposition 4.6 *Let $p,\tilde{p}\in(1,+\infty)$, $q,\tilde{q}\in[1,+\infty]$, $s\in(-1+\frac{1}{p},1+\frac{1}{p})$, $\alpha\in(-1+\frac{1}{\tilde{p}},1+\frac{1}{\tilde{p}})$ and $\lambda\in\Sigma_\mu$ provided $\mu\in[0,\pi)$. For $f\in\mathcal{S}'_h(\overline{\mathbb{R}^n_+},\mathbb{C})$, let us consider the* ***resolvent Neumann*** *problem with homogeneous boundary condition:*

$$\begin{cases}\lambda u-\Delta u &= f, \quad \text{in } \mathbb{R}^n_+,\\ \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} &= 0, \quad \text{on } \partial\mathbb{R}^n_+.\end{cases} \tag{$\mathcal{NL}_\lambda$}$$

Provided $(\mathcal{C}_{s,p})$ is satisfied, if $f\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, then the problem $(\mathcal{NL}_\lambda)$ admits a unique solution $u\in[\dot{\mathrm{H}}^{s,p}\cap\dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+,\mathbb{C})$ with estimate

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

If moreover $f\in\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$, then we also have $u\in[\dot{\mathrm{H}}^{\alpha,\tilde{p}}\cap\dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n_+,\mathbb{C})$ with corresponding estimate

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha,\mu} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)}.$$

The result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}})$ *by* $(\dot{\mathrm{B}}^{s}_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}})$ *whenever* $(\mathcal{C}_{s,p,q})$ *is satisfied.*

Proposition 4.7 *Let* $p, \tilde{p} \in (1,+\infty)$, $q, \tilde{q} \in [1,+\infty]$, $s \in (-1+\frac{1}{p}, +\infty)$, $\alpha \in (-1+\frac{1}{\tilde{p}}, +\infty)$ *such that* $(\mathcal{C}_{s+2,p})$ *is satisfied. For* $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C})$, *let us consider the* ***Dirichlet*** *problem with inhomogeneous boundary condition:*

$$\begin{cases} -\Delta u &= f, \quad \text{in } \mathbb{R}^n_+, \\ u_{|\partial\mathbb{R}^n_+} &= g, \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{$\mathcal{DL}_0$}$$

The problem $(\mathcal{DL}_0)$ *admits a unique solution* u *such that*

$$u \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+, \mathbb{C}) \subset \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C}))$$

with estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

If moreover $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+, \mathbb{C})$ *and* $g \in \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1}, \mathbb{C})$ *then the solution* u *also satisfies* $u \in \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}(\mathbb{R}^n_+, \mathbb{C})$ *with corresponding estimate*

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The result still holds if we replace both $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p})$ *and* $(\dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}, \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}})$ *by* $(\dot{\mathrm{B}}^{s}_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q})$ *and* $(\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{q}})$ *whenever* $(\mathcal{C}_{s+2,p,q})$ *is satisfied,* $q < +\infty$.

If $q = +\infty$, *everything still holds except* $x_n \mapsto u(\cdot, x_n)$ *is no more strongly continuous but only weak* continuous with values in* $\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}, \mathbb{C})$.

Proof. — Let $p \in (1,+\infty)$, $s > -1+1/p$, such that $(\mathcal{C}_{s+2,p})$ is satisfied. Then for $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C})$ we can write the problem $(\mathcal{DL}_0)$ as an evolution problem in the x_n variable,

$$\begin{cases} -\partial^2_{x_n} u - \Delta' u &= f, \quad \text{in } \mathbb{R}^{n-1} \times (0,+\infty), \\ u(\cdot,0) &= g, \quad \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{4.3}$$

Thanks to [ABHN11, Theorem 3.8.3], considering the semigroup $(e^{-x_n(-\Delta')^{1/2}})_{x_n \geqslant 0}$ and its mapping properties given by Proposition B.2 and Theorem 3.5, if $f = 0$, above problem admits unique solution $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C}))$. Thus, by linearity, we also have uniqueness of the solution u in $\mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C}))$ for non-identically zero function f. Therefore, it suffices to construct a solution.

Since $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, by definition, there exists $F \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n, \mathbb{C})$ such that

$$F_{|\mathbb{R}^n_+} = f, \qquad \text{and} \qquad \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \sim \|F\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Let $v := (-\Delta)^{-1} F \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n, \mathbb{C})$, we also have

$$\|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|F\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

So it suffices to prove the result for $w \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+, \mathbb{C})$, such that

$$\begin{cases} -\Delta w &= 0, \quad \text{in } \mathbb{R}^{n-1} \times (0,+\infty), \\ w_{|\partial\mathbb{R}^n_+} &= \tilde{g}, \quad \text{on } \mathbb{R}^{n-1}, \end{cases}$$

where $\tilde{g} \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C})$ can be seen as $g - v(\cdot,0)$. But such a w exists and is unique thanks to Proposition B.2 and [ABHN11, Theorem 3.8.3], and satisfies

$$\|w\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|\tilde{g}\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

Now, we can set $u := v + w$ which is a solution of $(\mathcal{DL}_0)$, and triangle inequality leads to

$$\begin{aligned}\|u\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} &\leqslant \|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} + \|w\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \\ &\lesssim_{p,n,s} \|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|v(\cdot,0)\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \\ &\lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\end{aligned}$$

which was the desired bound.

The Besov spaces case for $(f,g) \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\mathbb{C})\times\dot{\mathrm{B}}^{s+2-1/p}_{p,q}(\mathbb{R}^{n-1},\mathbb{C})$, whenever $(\mathcal{C}_{s+2,p,q})$ is satisfied, follows the same lines as before, except when $q=+\infty$ where the uniqueness argument can only be checked in a weak sense since $(e^{-x_n(-\Delta')^{1/2}})_{x_n\geqslant 0}$ is only weak* continuous in $\dot{\mathrm{B}}^{s+2-1/p}_{p,\infty}(\mathbb{R}^{n-1},\mathbb{C})$.

Now, if we assume that $f \in [\dot{\mathrm{H}}^{s,p}\cap\dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n_+,\mathbb{C})$ and $g \in [\dot{\mathrm{B}}^{s+2-1/p}_{p,p}\cap\dot{\mathrm{B}}^{\alpha+2-1/\tilde{p}}_{\tilde{p},\tilde{p}}](\mathbb{R}^{n-1},\mathbb{C})$, then with the same notations as above, by Proposition 3.8, we have

$$v = (-\Delta)^{-1}F \in [\dot{\mathrm{H}}^{s+2,p}\cap\dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n,\mathbb{C}) \text{ and } v(\cdot,0) \in \dot{\mathrm{B}}^{s+2-1/p}_{p,p}\cap\dot{\mathrm{B}}^{\alpha+2-1/\tilde{p}}_{\tilde{p},\tilde{p}}](\mathbb{R}^{n-1},\mathbb{C}).$$

From this, one may reproduce the estimates as above to obtain

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The case of intersection of Besov spaces follows the same lines. ■

We state the same result for the corresponding Neumann problem, for which the proof is very close.

Proposition 4.8 *Let* $p,\tilde{p} \in (1,+\infty)$, $q,\tilde{q} \in [1,+\infty]$, $s \in (-1+\frac{1}{p},+\infty)$, $\alpha \in (-1+\frac{1}{\tilde{p}},+\infty)$, *such that* $(\mathcal{C}_{s+2,p})$ *is satisfied. For* $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C})$, *let us consider the* ***Neumann*** *problem with inhomogeneous boundary condition:*

$$\begin{cases} -\Delta u &= f, \quad \text{in } \mathbb{R}^n_+, \\ \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} &= g, \quad \text{on } \partial\mathbb{R}^n_+. \end{cases} \qquad (\mathcal{NL}_0)$$

The problem $(\mathcal{NL}_0)$ *admits a unique solution* u *such that*

$$u \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+,\mathbb{C}) \subset \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}))$$

with estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

If moreover $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ *and* $g \in \dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1},\mathbb{C})$ *then the solution* u *also satisfies* $u \in \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ *with corresponding estimate*

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The result still holds replacing $(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{H}}^{s+2,p},\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p})$ *by* $(\dot{\mathrm{B}}^{s}_{p,q},\dot{\mathrm{B}}^{s+2}_{p,q},\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,q},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q})$ *and* $(\dot{\mathrm{H}}^{\alpha,\tilde{p}},\dot{\mathrm{H}}^{\alpha+2,\tilde{p}},\dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}})$ *by* $(\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}},\dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}},\dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{q}})$ *whenever* $(\mathcal{C}_{s+2,p,q})$ *is satisfied and* $q<+\infty$.

If $q=+\infty$, *everything still holds except* $x_n \mapsto u(\cdot,x_n)$ *is no more strongly continuous but only weak** *continuous with values in* $\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,\infty}(\mathbb{R}^{n-1},\mathbb{C})$.

We encourage the reader to compare with [DM15, Chapter 3].

A Complex interpolation for intersection of homogeneous Besov spaces

The next result is direct. Thanks to the fact that for all $a, b > 0$, $\theta \in [0,1]$,

$$a + a^{1-\theta}b^{\theta} \leqslant (a+b)^{\theta}a^{1-\theta} \leqslant 2^{\theta}(a + a^{1-\theta}b^{\theta}) \ ,$$

and since for $q \in [1,+\infty)$, $s_0, s_1 \in \mathbb{R}$, and for $\theta \in (0,1)$, if $s = (1-\theta)s_0 + \theta s_1$, we have with equivalence of norms

$$\ell^q_{s_0}(\mathbb{Z}) \cap \ell^q_{s}(\mathbb{Z}) = \ell^q(\mathbb{Z}, (2^{ks_0q} + 2^{ksq})\mathrm{d}k) = \ell^q(\mathbb{Z}, (2^{ks_0q} + 2^{ks_1q})^{\theta} 2^{ks_0q(1-\theta)}\mathrm{d}k) \ ,$$

therefore, by complex interpolation of weighted ℓ^q spaces, see [Tri78, Section 1.18.5], we obtain

Proposition A.1 *Let $q \in [1,+\infty)$, $s_0, s_1 \in \mathbb{R}$, consider a complex Banach space X, and for $\theta \in (0,1)$ let's introduce $s := (1-\theta)s_0 + \theta s_1$. The following equality holds with equivalence of norms*

$$[\ell^q_{s_0}(\mathbb{Z}, X), \ell^q_{s_0}(\mathbb{Z}, X) \cap \ell^q_{s_1}(\mathbb{Z}, X)]_{\theta} = \ell^q_{s_0}(\mathbb{Z}, X) \cap \ell^q_{s}(\mathbb{Z}, X) \ .$$

The result still holds with $\mathbb{N}$ instead of $\mathbb{Z}$.

Corollary A.2 *Let $p \in [1,+\infty]$, $q \in [1,+\infty)$, $s_j \in \mathbb{R}$, $j \in \{0,1\}$ such that $(\mathcal{C}_{s_0,p,q})$ is satisfied. Then for $\theta \in (0,1)$, let's introduce $s := (1-\theta)s_0 + \theta s_1$. Then the following equality holds with equivalence for norms*

$$[\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n)]_{\theta} = \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n) \ .$$

Proof. — Both function spaces $\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n)$, and $\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n)$ are complete normed vector spaces, see [BCD11, Theorem 2.25].

Now, we apply [BL76, Theorem 6.4.2] and Proposition A.1, claiming that, for all $s \in \mathbb{R}$, $\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)$ is a retraction of $\ell^q_s(\mathbb{Z}, \mathrm{L}^p(\mathbb{R}^n))$ through the homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$. ■

B Estimates for the Poisson semigroup

Lemma B.1 *Let $s > 0$, $\alpha \geqslant 0$ and $p, q \in [1,+\infty]^2$. For all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,*

$$\|u\|_{\dot{\mathrm{B}}^{\alpha-s}_{p,q}(\mathbb{R}^n)} \sim_{p,s,\alpha,n,q} \left\| t \mapsto \|t^s(-\Delta)^{\frac{\alpha}{2}} e^{-t(-\Delta)^{\frac{1}{2}}} u\|_{\mathrm{L}^p(\mathbb{R}^n)} \right\|_{\mathrm{L}^q_*(\mathbb{R}_+)} \ .$$

Proof. — It suffice to show the result for $\alpha = 0$. But in this case, the proof is straightforward the same as the one of [BCD11, Theorem 2.34] for the heat semigroup. ■

The following result was already proven in the case of homogeneous Besov spaces only. It is extended here to the case of homogeneous Sobolev spaces, we encourage the reader to compare with [DM09, Lemma 2].

Proposition B.2 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$. The map*

$$T \,:\, f \longmapsto \left[(x', x_n) \mapsto e^{-x_n(-\Delta')^{\frac{1}{2}}} f(x') \right]$$

is such that

(i) Given $s \geqslant 0$, for all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \ .$$

In particular, T extends uniquely as a bounded linear operator $T \,:\, \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p})$ is satisfied.

(ii) *Given* $s>0$, *for all* $f\in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$, *we have*

$$\|Tf\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}\,.$$

In particular, T *extends uniquely as a bounded linear operator* $T:\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})\longrightarrow\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)$ *whenever* $(\mathcal{C}_{s,p,q})$ *is satisfied.*

Proof. — **Point** ***(i)*:** For $p\in(1,+\infty)$, let's consider $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$. We apply Lemma B.1 to obtain,

$$\begin{aligned}\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} &= \left(\int_0^{+\infty}\|e^{-x_n(-\Delta')^{\frac{1}{2}}}f\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})}\,\mathrm{d}x_n\right)^{\frac{1}{p}}\\ &= \left(\int_0^{+\infty}\left(t^{\frac{1}{p}}\|e^{-t(-\Delta')^{\frac{1}{2}}}f\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}\\ &\lesssim_{p,n} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.\end{aligned}$$

We continue noticing that for all $f\in\mathcal{S}'_h(\mathbb{R}^{n-1})$, $m\in\mathbb{N}$, $\partial^m_{x_n}Tf=(-\Delta')^{\frac{m}{2}}Tf=T(-\Delta')^{\frac{m}{2}}f$ and $Tf\in\mathcal{S}'_h(\mathbb{R}^{n-1})$, thus if $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ we may apply previous inequality to obtain,

$$\begin{aligned}\|Tf\|_{\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n_+)} &\sim_{p,n,m} \|\partial^m_{x_n}Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+\|(-\Delta')^{\frac{m}{2}}Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)}\\ &\sim_{p,n,m} \|T(-\Delta')^{\frac{m}{2}}f\|_{\mathrm{L}^p(\mathbb{R}^n_+)}\\ &\lesssim_{p,n,m} \|f\|_{\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.\end{aligned}$$

So that for all $m\in\mathbb{N}$, all $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{m,p}(\mathbb{R}^n_+)} \lesssim_{p,n,m} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Thus, by complex interpolation and Corollary A.2, for all $s\geqslant 0$, all $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Hence, thanks to Proposition 2.18, $\mathrm{H}^{s,p}(\mathbb{R}^n_+)=\mathrm{L}^p(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$,

$$\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Therefore, if $\lambda\in 2^{\mathbb{N}}$, we can consider f_λ is the dilation by factor λ of f, so that plugging f_λ instead of f in above inequality, and checking the fact that $Tf_\lambda=(Tf)_\lambda$, we obtain

$$\lambda^{-\frac{n}{p}}\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+\lambda^{s-\frac{n}{p}}\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \lambda^{-\frac{n}{p}}\|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\lambda^{s-\frac{n}{p}}\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

One may divide above inequality by $\lambda^{s-\frac{n}{p}}$, so that letting λ grow to infinity yields

$$\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

So that the result holds by density whenever $(\mathcal{C}_{s,p})$ is satisfied.

Point ***(ii)*:** Now let $q\in[1,+\infty]$, since for all $s\geqslant 0$, all $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Hence, by real interpolation, using [Haa06, Proposition B.2.7] instead of Corollary A.2, we obtain that for all $s>0$ all $f\in\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{B}^{s}_{p,q}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}\,.$$

Then the same dilation procedure as before, yields

$$\|Tf\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \text{ ,}$$

which again allows to conclude via density argument if $q<+\infty$ and ($\mathcal{C}_{s,p,q}$) is satisfied. The case $q=+\infty$, when ($\mathcal{C}_{s,p,q}$) is satisfied, follows from real interpolation with the last estimate. ■

Proposition B.2 can be self-improved as

Corollary B.3 *Let $p_j\in(1,+\infty)$, $q_j\in[1,+\infty)$, $j\in\{0,1\}$. The map*

$$T\,:\,f\longmapsto\left[(x',x_n)\mapsto e^{-x_n(-\Delta')^{\frac{1}{2}}}f(x')\right]$$

is such that

(i) Let $s_j\geqslant 0$, $j\in\{0,1\}$, such that ($\mathcal{C}_{s_0,p_0}$) is satisfied. For all $f\in[\dot{\mathrm{B}}^{s_0-\frac{1}{p_0}}_{p_0,p_0}\cap\dot{\mathrm{B}}^{s_1-\frac{1}{p_1}}_{p_1,p_1}](\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})}\text{ , } j\in\{0,1\}\text{ .}$$

(ii) Let $s_j>0$, $j\in\{0,1\}$, such that ($\mathcal{C}_{s_0,p_0,q_0}$) is satisfied. For all $f\in[\dot{\mathrm{B}}^{s_0-\frac{1}{p_0}}_{p_0,q_0}\cap\dot{\mathrm{B}}^{s_1-\frac{1}{p_1}}_{p_1,q_1}](\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})}\text{ , } j\in\{0,1\}\text{ .}$$

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