Are the Liquidity and Collateral Roles of Asset Bubbles Different?

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Several papers explain why asset bubbles are observed when growth is large. These papers differ in the role of the bubble, used to provide liquidities or as collateral in a borrowing constraint. We compare the liquidity and collateral roles of bubbles in an overlapping generations model. When the bubble is deterministic, the equilibrium is identical under these two roles, implying that the same mechanism explains the crowding-in effect of the bubble on growth. With stochastic bubbles, growth is larger when bubbles play the liquidity role, because the burst of a bubble used for liquidity is less damaging to capital investors.

JEL codes: D15, E44, G11
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The financial crises of recent years have led to a renewed interest in the study of the interplay between the financial and real spheres of the economy. In particular, several contributions show that episodes of speculative bubbles are associated to periods of economic expansions and bubble crashes are as-

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sociated to recessions (see Caballero, Farhi, and Hammour 2006, Martin and Ventura 2012, Brunnermeier and Oehmke 2013, Kindleberger and Aliber 2015). Explaining this evidence is challenging because seminal papers show that the existence of rational bubbles in dynamic general equilibrium models is associated to lower GDP per capita (Tirole 1985) or growth Grossman and Yanagawa (1993). This is the so-called crowding-out effect of the bubble.

Most of the papers that reconcile the existence of rational bubbles with the empirical facts introduce financial imperfections embodied in borrowing constraints (see Miao 2014 for a short survey) and heterogeneous agents to have different types of traders on the asset markets. The recent and growing literature about rational bubbles with financial frictions distinguishes between two growth-enhancing roles of the bubbles or, equivalently, crowding-in effects. One is the liquidity role of the bubble: agents hold at the beginning of the period the bubble and sell it to increase their productive investment (Caballero and Krishnamurthy 2006, Farhi and Tirole 2012, Hirano and Yanagawa 2017, Kocherlakota 2009, Kiyotaki and Moore 2019, Martin and Ventura 2012, Miao and Wang 2018).[1] The other one is the collateral role of the bubble: agents buy the bubble to increase their possibilities to borrow and use these loans to invest in capital (Bengui and Phan 2018, Kocherlakota 2009, Miao, Wang, and Zhou 2015, Martin and Ventura 2016, Miao and Wang 2018).

There are different ways to introduce heterogeneous traders. For instance, Kocherlakota (2009), Martin and Ventura (2012), Hirano and Yanagawa (2017), Kiyotaki and Moore (2019) consider heterogeneous agents born at the same period, unproductive versus productive ones. Another possibility is to introduce entrepreneurs facing idiosyncratic shocks (Miao and Wang 2018, Miao, Wang, and Zhou 2015). Heterogeneity of traders can also be introduced using overlapping generations with agents living three periods, as in several recent papers like Arce and Lopez-Salido (2011), Farhi and Tirole (2012), Basco (2014, 2016), or Raurich and Seegmuller (2019), among others. In these contributions, heterogeneity is among agents born at different periods. When agents are heterogeneous and face credit market imperfections, bubbles can channel liquidities from unproductive agents (lenders–savers) to productive ones (borrowers–investors), which is essential for the existence of the crowding-in effect of the bubble.

In this paper, we study the differences between the liquidity and collateral roles of the bubble. Our purpose is to contribute to the literature on bubbles by identifying the mechanisms behind the crowding-in effect of a bubble. We especially compare the liquidity and collateral roles when the bubble is stochastic.

1. Note that this liquidity role has also been emphasized in a different perspective by Woodford (1990). Instead of being concerned with bubbles, he focuses on nonneutrality of public debt. In their paper, Kiyotaki and Moore (2019) are interested in the liquidity role of fiat money, which can be seen as a rational bubble. Fiat money allows unproductive entrepreneurs to transfer some liquidities toward productive ones, who have an investment opportunity.

We compare the two roles of the bubble in a three-period lived agents model.[2] We distinguish among three types of traders (young, adult, and old), while only two of them can buy assets and invest in capital (young and adult). To introduce heterogeneous traders, we assume that young households cannot invest in capital, while adults invest in this asset expecting a positive return. It means that adults are the most and only productive investors. At each period of time, there is a credit market in which young households and adults can save and borrow. The amount of credit is limited by a borrowing constraint. There is also a stochastic bubble, which faces a positive probability of market crash at each period of time.

We consider two models in which, at the second period of life, borrowing is constrained and collateralized by capital, that is, a fundamental collateral. In the first one, the bubble is bought by young households and sold by adults. Therefore, adults can sell the bubble to invest more in capital. Selling the bubble corresponds to a transfer from the unproductive young agents to productive adults. This mechanism extends to a general equilibrium framework the liquidity effect of the bubble developed in Farhi and Tirole (2012) and it is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009), Martin and Ventura (2012), or Miao and Wang (2018). In the second model, following Kocherlakota (2009), Martin and Ventura (2016), or Miao and Wang (2018), the bubble is only bought by adults and used as a collateral in the borrowing constraint. By increasing the collateral, the bubble increases the amount borrowed, promoting a higher investment in capital. These two models illustrate the two different roles of bubbles. To fix ideas and to be able to analyze the dynamics in a simple way, firms produce the final good using an Ak technology, which implies endogenous growth.

We start by considering that the probability of bubble crash is zero, that is, the bubble is deterministic. In this case, we show that the two models lead to exactly the same equilibrium, despite the fact that the two mechanisms of the bubble seem to be *a priori* different. This has been shown by Miao and Wang (2018) in a model with infinitely lived agents. We show that this result also holds in an overlapping generations model. Since the borrowing constraint is binding, capital is not perfectly substitutable with the two other assets and households cannot smooth consumption perfectly between adult and old ages. The bubble promotes investment and has a positive effect on growth, whereas the resulting increase in the interest rate has a negative effect when capital is used as collateral. When the degree of pledgeability of the fundamental collateral is small enough, the first effect dominates and the bubble enhances growth, that is, has a crowding-in effect. On the contrary, when the degree of pledgeability is sufficiently large, the bubble has a crowding-out effect on growth. These results do not depend on the particular type of bubble considered. When the

2. We argue that our results are not dependent of the type of heterogeneity we consider. To illustrate this point, in an Online Appendix, we construct a model with heterogenous infinitely lived agents and show that we obtain some results that are comparable to those obtained when heterogeneity comes from an OLG model with agents living three periods.

bubble is deterministic, there is no distinction between the liquidity and the collateral roles of the bubble.

We finally focus on the stochastic case where bubble burst occurs with a positive probability (Weil 1987). The liquidity and collateral roles of a bubble can again be compared and we show that, in contrast to the deterministic case, there is a difference between the two roles. When a household buys the bubble for its liquidity role at the young age, she faces more risk in terms of consumption than when she buys it for its collateral role at her adult age. Therefore, the bubble has a lower size when it has a liquidity role. The opposite conclusion applies for capital investment and growth, which are larger when the bubble has the liquidity role. A bubble used for its liquidity role generates a higher growth than a bubble used as a collateral, because the possible bubble crash is less damaging to agents who invest in capital. These new results show that when the bubble is stochastic, the liquidity and collateral roles are no more equivalent and identical. This has never been emphasized in the literature. Despite their quantitative difference, the liquidity and collateral roles of the bubble are comparable since both of them have a crowding-in effect. Furthermore, we show that the crowding-in effect of both types of bubbles is similar when the degree of pledgeability of capital is small.

In the following section, we present the two models in which the stochastic bubble is either bought when young and used to provide liquidities or used as a collateral when adult. In Section 2, we compare and analyze the models when the bubble is deterministic. In Section 3, we make the comparison between the liquidity and collateral roles when the bubble is stochastic. Section 4 concludes and technical details are relegated to the Appendix.

1. TWO MODELS WITH LIQUIDITY AND COLLATERAL ROLES OF BUBBLES

As in several recent papers (Arce and Lopez-Salido 2011, Basco 2014, 2016, Farhi and Tirole 2012, Raurich and Seegmuller 2019), we consider an overlapping generations model with three-period lived agents. Therefore, agents may invest both when young or adult. We distinguish between savers and investors. The former only save through financial assets, while the later also invest in productive capital (Farhi and Tirole 2012).[3] This distinction introduces heterogeneity since adults are the only productive investors. We also assume that adult individuals face a borrowing constraint. This framework allows us to consider the two roles of the bubble mentioned above. In the first model we present, the bubble has a liquidity role when an adult household sells the bubble bought when young to invest in capital. In the second model, the bubble is bought by adult agents, and it plays the role of a collateral in the borrowing

3. Note that in a recent paper, Raurich and Seegmuller (2019) already investigate the economy where the young invest in capital, while adults do not have access to the capital market.

constraint.[4] To fix ideas, we start by presenting the production sector which will be the same whatever the model we will consider.

1.1 Production Sector

To simplify the dynamic analysis, we introduce a simple Ak technology. Aggregate output is produced by firms, using labor, l_t, and capital, k_t, as inputs. In addition, production benefits from an externality that summarizes a learning-by-doing process, and allows to have sustained growth. Following Frankel 1962) or (Ljungqvist and Sargent (2004, ch. 14), this externality depends on the average capital–labor ratio.

Letting $a_t \equiv k_t/l_t$, $\overline{a}_t$ represents the average ratio of capital over labor. Firms produce the final good using the following technology:

$$y_t = F(k_t, \bar{a}_t l_t).$$

The technology $F(k_t, \bar{a}_t l_t)$ has the usual neoclassical properties, that is, a strictly increasing and concave production function satisfying the Inada conditions, and is homogeneous of degree one with respect to its two arguments.

Profit maximization under perfect competition implies that the wage w_t and the return of capital q_t are given by[5]:

$$w_t = F_2(k_t, \bar{a}_t l_t)\bar{a}_t, \tag{1}$$

$$q_t = F_1(k_t, \bar{a}_t l_t). \tag{2}$$

All equilibria we will consider are symmetric ones, that is, $a_t = \overline{a}_t$. Let us define $s \equiv F_1(1, 1)/F(1, 1) \in (0, 1)$ the capital share in total production and $A \equiv F(1, 1) > 0$. Using (1) and (2), we deduce that:

$$w_t = (1 - s)Aa_t \equiv w(a_t), \tag{3}$$

$$q_t = sA = q, \tag{4}$$

which give the wage and the return of capital at an equilibrium.

1.2 Model with Bubble Bought by Young Savers, YS

In this first model, that we denote as YS because young savers buy the bubble, we illustrate the liquidity role of the bubble. Agents buy the bubble when they are

4. In this paper, agents that trade the different assets are identified as households, whereas some papers rather speak about entrepreneurs (Farhi and Tirole 2012, Hirano and Yanagawa 2017, Kocherlakota 2009, Miao and Wang 2018). The difference only concerns the denomination of the agents.

5. We denote by $F_i(., .)$ the derivative with respect to the ith argument of the function.

savers and they sell it to increase investment in capital when they become investors. In this way, the bubble promotes growth. This crowding-in mechanism is the liquidity role of the bubble that was introduced in Farhi and Tirole (2012) in a partial equilibrium framework. The mechanism behind this example is also in line with many other existing papers like Hirano and Yanagawa (2017), Kocherlakota (2009), Martin and Ventura (2012), or Miao and Wang (2018).

We consider an overlapping generations economy populated by agents living for three periods. An agent is young in the first period of life, adult in the second period, and old in the third period. There is no population growth. The population size of a generation is constant and normalized to one.

Each household derives utility from consumption at each period of time. Preferences of an individual born in period t are represented by the following expected utility function:

$$\mathbb{E}_t[\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2})], \tag{5}$$

where $\alpha, \beta, \gamma > 0$ and $u_i(c_{it}) = \ln c_{it}$.

The household inelastically supplies one unit of labor when young and adult. When young, labor efficiency is one, while it is equal to $\phi > 0$ when adult. There are three assets in the economy: capital k_t used in the production, deposits d_{it} that allow to finance loans, and an asset without fundamental value supplied in one unit, with a price b_{1t}. There is a bubble as soon as $b_{1t} > 0$.

Following the seminal paper by Weil (1987), we assume that households may coordinate their expectations in an equilibrium where the bubble crashes, that is, its value is zero in the next period. Because of the volatility of agents' expectations, there is a positive probability of bubble crash in each period of time. We consider a Markov process of a bubble crash. If there is no bubble at period t, there is no bubble at period $t+1$ with a probability equal to one. If there is a bubble at period t, there is a probability $\pi \in (0, 1]$ such that the bubble persists in the next period and a probability $1 - \pi$ such that the bubble crashes at period $t+1$.

When young, the household saves through deposits d_{1t} and can buy the bubble b_{1t}. In the next period, these two assets provide returns given by R^d_{t+1} and $r_{t+1} > 0$ if there is no bubble crash. On the contrary, a market crash in period $t+1$ means that the return of the asset without fundamental value is zero, that is, $r_{t+1} = 0$. When adult, the household can only save through deposits d_{2t+1} and invests in capital k_{t+2}. When old, these two assets are remunerated with the returns R^d_{t+2} and q_{t+2}, respectively. Of course, when $d_{it} < 0$, the household rather contracts loans. When adult, these loans are limited by the following borrowing constraint:

$$-R^d_{t+2} d_{2t+1} \leqslant \theta q_{t+2} k_{t+2}, \tag{6}$$

where $\theta \in [0, 1)$ is the degree of pledgeability. This constraint means that, when adult, the household can borrow an amount $d_{2t+1} < 0$, as long as the repayment does not exceed a fraction θ of the future return from her productive investment at period $t+2$.

The parameter θ determines the financial market imperfection, where a lower θ means a stronger imperfection. Young agents will not face an equivalent constraint, because they will not be short sellers of the liquid assets, that is, deposits and the bubble, but rather use them to make transfers to the next periods of life.

Let us assume that a bubble exists at period t when the household is young.[6] At this period, her budget constraint is given by:

$$c_{1t} + d_{1t} + b_{1t} = w_t. \tag{7}$$

With probability π the bubble persists in the next period and the budget constraint when adult is given by[7]:

$$c^{+}_{2t+1} + k_{t+2} + d_{2t+1} = \phi w_{t+1} + r_{t+1} b_{1t} + R^{d}_{t+1} d_{1t}, \tag{8}$$

whereas it is given by[8]:

$$c^{0}_{2t+1} + k^{0}_{t+2} + d^{0}_{2t+1} = \phi w_{t+1} + R^{d}_{t+1} d_{1t}, \tag{9}$$

when there is a crash of the bubble, which occurs with probability $1 - \pi$. At the old age, the budget constraint is not directly affected by the value of the bubble, but depends on what happens in the previous period. Therefore, we distinguish between:

$$c^{+}_{3t+2} = q_{t+2} k_{t+2} + R^{d}_{t+2} d_{2t+1}, \tag{10}$$

$$c^{0}_{3t+2} = q_{t+2} k^{0}_{t+2} + R^{d0}_{t+2} d^{0}_{2t+1}. \tag{11}$$

Whatever the state of the nature, the borrowing constraint (6) is assumed to be binding, that is, $R^{d}_{t+2} d_{2t+1} = -\theta q_{t+2} k_{t+2}$ and $R^{d0}_{t+2} d^{0}_{2t+1} = -\theta q_{t+2} k^{0}_{t+2}$.[9] We focus on such type of equilibria to be in accordance with the literature, as for instance Farhi and Tirole (2012) and Hirano and Yanagawa (2017). Considering a binding borrowing

6. When there is no bubble at period t, the economy is at the bubbleless equilibrium for all the following periods.

7. We introduce the superscript $+$ for the consumptions when adult and old if the bubble does not crash to make a difference with the consumptions that appear in the expected utility (5).

8. Note that in what follows, a superscript 0 is added to each variable when it corresponds the situation where the bubble has crashed.

9. From the first-order condition of the household program, we could check that these borrowing constraints are binding under similar conditions than in the deterministic model, that is, $q_{t+2} > R^{d}_{t+2} > \theta q_{t+2}$ and $q_{t+2} > R^{d0}_{t+2} > \theta q_{t+2}$. Note that the equilibrium cannot satisfy $R^{d}_{t+2} > q_{t+2}$ ($R^{d0}_{t+2} > q_{t+2}$) because, in this case, adults will not invest in capital since d_{2t+1} (d^{0}_{2t+1}) gives a higher return. Note also that $R^{d}_{t+2} < \theta q_{t+2}$ ($R^{d0}_{t+2} < \theta q_{t+2}$) cannot occur as it would imply that an adult could borrow an infinite amount to invest in capital without being constrained.

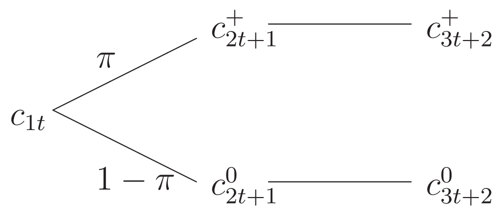

Fig 1. Consumption Profile in the Model YS when $1-\pi$ is the Probability of Bubble Crash.

constraint in each state of nature will also facilitate the comparison with the deterministic model where the probability of market crash is zero. Using (8), (9) and these constraints, we deduce that:

$$c^+_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t} - k_{t+2}\left(1 - \theta \frac{q_{t+2}}{R^d_{t+2}}\right), \tag{12}$$

$$c^0_{2t+1} = \phi w_{t+1} + R^d_{t+1} d_{1t} - k^0_{t+2}\left(1 - \theta \frac{q_{t+2}}{R^{d0}_{t+2}}\right). \tag{13}$$

In accordance with Farhi and Tirole (2012) and Hirano and Yanagawa (2017), R^d_{t+1} is the same in the two states of nature. When there is a bubble at period t, using the equilibrium on the debt market $d_{1t} + d_{2t} = 0$ and the binding borrowing constraint, we get $R^d_{t+1} d_{1t} = \theta q_{t+1} k_{t+1}$, where $q_{t+1} = sA$, and d_{1t} and k_{t+1} are determined at period t. This explains that R^d_{t+1} is uniquely defined in both equations (12) and (13).[10]

Since the consumption when old does not depend on the value of the bubble, the consumption will be c^+_{3t+2} when the consumption was c^+_{2t+1} at middle age and c^0_{3t+2} when the consumption was c^0_{2t+1} at middle age. Substituting the binding borrowing constraints into (10), we obtain:

$$c^+_{3t+2} = (1-\theta) q_{t+2} k_{t+2}, \tag{14}$$

$$c^0_{3t+2} = (1-\theta) q_{t+2} k^0_{t+2}. \tag{15}$$

Considering these events (see also Figure 1), the household's expected utility writes:

$$\begin{aligned}\mathbb{E}_t[\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2})] &= \alpha u_1(c_{1t}) \\ +\beta\left[\pi u_2(c^+_{2t+1}) + (1-\pi) u_2(c^0_{2t+1})\right] &+ \gamma\left[\pi u(c^+_{3t+2}) + (1-\pi) u_3(c^0_{3t+2})\right].\end{aligned} \tag{16}$$

10. This would also arise if we would rather consider d_{1t} as debt contracts without risk.

A household maximizes this utility function with respect to d_{1t}, b_{1t}, k_{t+2}, and k^0_{t+2} under (7) and (12)–(15). Using $u_i(c_{it}) = \ln c_{it}$, we obtain:

$$\frac{\alpha}{c_{1t}} = R^d_{t+1}\beta\left(\frac{\pi}{c^+_{2t+1}} + \frac{1-\pi}{c^0_{2t+1}}\right), \tag{17}$$

$$\frac{\alpha}{c_{1t}} = r_{t+1}\beta\frac{\pi}{c^+_{2t+1}}, \tag{18}$$

$$\frac{\beta}{c^+_{2t+1}}\left(1-\theta\frac{q_{t+2}}{R^d_{t+2}}\right) = (1-\theta)q_{t+2}\frac{\gamma}{c^+_{3t+2}}, \tag{19}$$

$$\frac{\beta}{c^0_{2t+1}}\left(1-\theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) = (1-\theta)q_{t+2}\frac{\gamma}{c^0_{3t+2}}. \tag{20}$$

Finally, when it exists, the bubble evolves according to:

$$b_{1t+1} = r_{t+1}b_{1t}, \tag{21}$$

where r_{t+1} also measures the growth of the bubble price b_{1t+1}/b_{1t}.

As it is shown in Appendix A.1, using the equilibrium prices (3) and (4), the equilibrium on the debt market $d_{1t} = -d_{2t} = \theta q_{t+1}k_{t+1}/R^d_{t+1}$, the equilibrium on the labor market $l_t = 1+\phi$, the capital $k_t = a_t(1+\phi)$, the growth factor $g_{t+1} = a_{t+1}/a_t$, and bubble per unit of capital $\widetilde{b}_{1t} = b_{1t}/[(1+\phi)a_t]$, we can define an equilibrium as a sequence $\{\widetilde{b}_{1t}, g_t\}_{t\geqslant 0}$ satisfying the following two equations:

$$\begin{aligned} g_{t+1} = {} & \frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)} + \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right] \\ & +\widetilde{b}_{1t}\left[\frac{\gamma}{\beta+\gamma} - \theta s\frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right], \end{aligned} \tag{22}$$

$$\widetilde{b}_{1t+1} = \frac{\theta sA\left(1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right)}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{\pi(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left[\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right]}\widetilde{b}_{1t} \equiv I(\widetilde{b}_{1t}) \tag{23}$$

1.3 *Model with Bubble Bought by Adult Investors, AI*

This model, that we call AI because adult investors buy the bubble, illustrates the collateral role of the bubble, introduced by Kocherlakota (2009), Martin and Ventura

(2016), or Miao and Wang (2018). Investors buy the bubble to use it as collateral for credits that finance investment in capital.

The model is the same as the YS, except for the bubble. There is still an asset without fundamental value supplied in one unit, but it is not bought by young households and sold by adults. Instead, it is bought by adults at the price b_{2t+1} and sold by old households, with a return equal to R_{t+2}. We consider the behavior of a household born at period t, assuming that a bubble exists at that period. Households do not buy the bubble when young, which implies that the budget constraint is given by:

$$c_{1t} + d_{1t} = w_t. \tag{24}$$

Depending on whether the bubble has crashed or not at the beginning of period $t+1$, the bubble is bought or not at middle age. The consumptions at middle age associated to these two events are c^{+}_{2t+1} and c^{0}_{2t+1}, respectively. The associated budget constraints are given by:

$$c^{+}_{2t+1} + k_{t+2} + d_{2t+1} + b_{2t+1} = \phi w_{t+1} + R^{d}_{t+1} d_{1t}, \tag{25}$$

$$c^{0}_{2t+1} + k^{0}_{t+2} + d^{0}_{2t+1} = \phi w_{t+1} + \widetilde{R}^{d}_{t+1} d_{1t}. \tag{26}$$

In contrast to the model YS with a stochastic bubble, the return of d_{1t} is no more the same in the two states of nature. R^{d}_{t+1} is still the return of loans when the bubble persists, but $\widetilde{R}^{d}_{t+1}$ is the return when the bubble just crashes in $t+1$. In addition, as in the model YS, R^{d0}_{t+1} denotes the return of loans when there is no bubble.

If the bubble has crashed at middle age, there is no bubble at the following periods, which implies that the consumption when old is:

$$c^{00}_{3t+2} = q_{t+2} k^{0}_{t+2} + R^{d0}_{t+2} d^{0}_{2t+1}. \tag{27}$$

If the bubble has not crashed at middle age, the consumption when old depends on whether the bubble crashes (c^{+0}_{3t+2}) or not (c^{++}_{3t+2}) at old age:

$$c^{+0}_{3t+2} = q_{t+2} k_{t+2} + \widetilde{R}^{d}_{t+2} d_{2t+1}, \tag{28}$$

$$c^{++}_{3t+2} = q_{t+2} k_{t+2} + R^{d}_{t+2} d_{2t+1} + R_{t+2} b_{2t+1}. \tag{29}$$

As in YS model, adult investors face a borrowing constraint. We assume that this binds in every state of nature. We distinguish between two states. First, when the bubble has crashed at middle age, we assume that the borrowing constraint

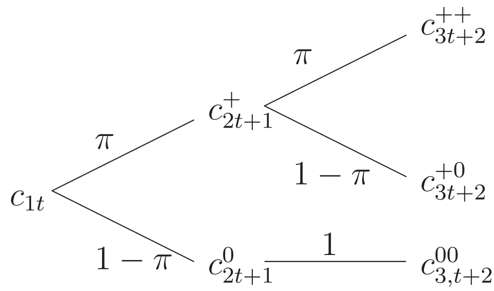

Fig 2. Consumption Profile in the Model AI when $1-\pi$ is the Probability of Bubble Crash.

$-R^{d0}_{t+2}d^0_{2t+1} \leqslant \theta q_{t+2}k^0_{t+2}$ is binding, that is, $R^{d0}_{t+2}d^0_{2t+1} = -\theta q_{t+2}k^0_{t+2}$. Then, the consumptions (26) and (27) rewrite:

$$c^0_{2t+1} = \phi w_{t+1} + \widetilde{R}^d_{t+1}d_{1t} - k^0_{t+2}\left(1-\theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right), \tag{30}$$

$$c^{00}_{3t+2} = (1-\theta)q_{t+2}k^0_{t+2}. \tag{31}$$

Second, when there is a bubble at the middle age, $b_{2t+1} > 0$, we follow Martin and Ventura (2016) assuming that the bubble is also used as collateral to borrow. We have now both fundamental and bubbly collaterals. The credit constraint is stochastic, that is, it depends on the expected return of the bubble and on the expected cost of credit reimbursement.[11] We have:

$$-\left[\pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2}\right]d_{2t+1} \leqslant \theta q_{t+2}k_{t+2} + \pi R_{t+2}b_{2t+1}. \tag{32}$$

The household can borrow an amount $d_{t+2} < 0$, as long as the expected repayment does not exceed a fraction θ of the future return from her productive investment, that is, the fundamental collateral, and the expected market value of the bubble at period $t+2$, that is, the bubbly collateral. Note that the degree of pledgeability of the bubbly collateral is one. This is in accordance with Kocherlakota (2009), Martin and Ventura (2016), or Miao and Wang (2018). In fact, Kocherlakota (2009) introduces a constraint where only the bubble serves as collateral. It corresponds to the case where $\theta = 0$.

Considering the consumptions at the different states (see also Figure 2), a household maximizes her expected utility:

$$\begin{aligned}\mathbb{E}_t[\alpha u_1(c_{1t}) + \beta u_2(c_{2t+1}) + \gamma u_3(c_{3t+2})] &= \alpha u_1(c_{1t}) + \beta[\pi u_2(c^+_{2t+1}) \\ +(1-\pi)u_2(c^0_{2t+1})]+\gamma[\pi^2 u(c^{++}_{3t+2})&+\pi(1-\pi)u_3(c^{+0}_{3t+2}) + (1-\pi)u_3(c^{00}_{3t+2})]\end{aligned}$$

11. See, for instance, Iacovello (2015) and Quadrini (2011) who, in a different context, also consider borrowing constraints in expected terms.

with respect to d_{1t}, k^0_{t+2}, k_{t+2}, b_{2t+1}, and d_{2t+1} under the constraints (24), (25), and (28)–(32). From the first-order condition with respect to d_{1t}, we obtain the arbitrage condition between consumptions when young and adult:

$$\frac{\alpha}{c_{1t}} = \frac{\beta\pi R^d_{t+1}}{c^+_{2t+1}} + \frac{\beta(1-\pi)\widetilde{R}^d_{t+1}}{c^0_{2t+1}}. \tag{33}$$

If the bubble has crashed at period $t+1$, we obtain from the first-order condition with respect to k_{0t+2}:

$$c^0_{2t+1} = \frac{\beta}{\beta+\gamma}\left(\phi w_{t+1} + \widetilde{R}^d_{t+1} d_{1t}\right), \tag{34}$$

$$k^0_{t+2}\left(1 - \theta\frac{q_{t+2}}{R^{d0}_{t+2}}\right) = \frac{\gamma}{\beta+\gamma}\left(\phi w_{t+1} + \widetilde{R}^d_{t+1} d_{1t}\right). \tag{35}$$

If the bubble has not crashed at period $t+1$, we assume that the borrowing constraint (32) is binding and we deduce the following first-order conditions[12]:

$$\left(\pi - \theta\frac{q_{t+2}}{R_{t+2}}\right)\frac{\beta}{c^+_{2t+1}} = q_{t+2}\gamma\left(\frac{\pi(\pi-\theta)}{c^{++}_{3t+2}} + \frac{\pi(1-\pi)}{c^{+0}_{3t+2}}\right), \tag{36}$$

$$\frac{\beta}{c^+_{2t+1}}\left(\pi - \frac{\pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2}}{R_{t+2}}\right) = \frac{\gamma\pi(1-\pi)\widetilde{R}^d_{t+2}}{c^{+0}_{3t+2}} - \frac{\gamma\pi(1-\pi)\widetilde{R}^d_{t+2}}{c^{++}_{3t+2}} \tag{37}$$

We assume that the asset d_{2t+1} is a financial instrument that gives perfect insurance in terms of consumption, that is, $c^{++}_{3t+2} = c^{+0}_{3t+2}$. Using the expected borrowing constraint (32), the budget constraints (28) and (29) allow us to deduce that $c^{++}_{3t+2} = c^{+0}_{3t+2} = (1-\theta)q_{t+2}k_{t+2}$. Such an assumption, which is in accordance with Kocherlakota (2009), will facilitate the comparison with the model where the bubble has a liquidity role. Moreover, we will obtain the deterministic model as a limit case when π tends to one, which is not possible otherwise.

Using (37), we immediately deduce that:

$$\pi R_{t+2} = \pi R^d_{t+2} + (1-\pi)\widetilde{R}^d_{t+2}. \tag{38}$$

12. Using the first-order conditions of the household problem, we can easily show that the borrowing constraint is binding when $q_{t+2} > \pi R_{t+2} > \theta q_{t+2}$.

Therefore, the borrowing constraint (32) rewrites:

$$d_{2t+1} = -\frac{\theta q_{t+2} k_{t+2}}{\pi R_{t+2}} - b_{2t+1}. \tag{39}$$

The equilibrium on the credit market implies that $d_{1t+1} = -d_{2t+1}$. Using these different equilibrium conditions and the budget constraint (25), the first-order condition (36) implies:

$$c^{+}_{2t+1} = \frac{\beta}{\beta+\gamma}\big(\phi w_{t+1} + R^d_{t+1} d_{1t}\big), \tag{40}$$

$$k_{t+2}\left(1-\theta\frac{q_{t+2}}{\pi R_{t+2}}\right) = \frac{\gamma}{\beta+\gamma}\big(\phi w_{t+1} + R^d_{t+1} d_{1t}\big). \tag{41}$$

The bubble evolves according to:

$$b_{2t+1} = R_{t+1} b_{2t}. \tag{42}$$

Using the equilibrium on the credit market and (38), we obtain $R^d_{t+1} d_{1t} = -d_{2t}(R_{t+2} - \frac{1-\pi}{\pi}\widetilde{R}^d_{t+2})$. Substituting the equality $-\widetilde{R}^d_{t+2} d_{2t+1} = \theta q_{t+2} k_{t+2}$, which comes from the perfect consumption insurance, and (39), we obtain $R^d_{t+1} d_{1t} = b_{2t+1} R_{t+2} + \theta q_{t+2} k_{t+2}$. Using this and (42), equation (41) rewrites one period before as:

$$k_{t+1}\left(1-\theta\frac{q_{t+1}}{\pi R_{t+1}}\right) = \frac{\gamma}{\beta+\gamma}(\phi w_t + b_{2t} + \theta q_t k_t). \tag{43}$$

Let us introduce $a_t \equiv k_t/(1+\phi)$, $\widetilde{b}_{2t} \equiv b_{2t}/[(1+\phi)a_t]$, $g_{t+1} \equiv a_{t+1}/a_t$, and:

$$\Psi(\widetilde{b}_{2t+1}) \equiv \frac{\theta sA\left(\frac{\phi(1-s)A}{1+\phi} + \theta sA\right) + \widetilde{b}_{2t+1}\left(\frac{\pi\phi(1-s)A}{1+\phi} + \theta sA\right)}{\left(\frac{\phi(1-s)A}{1+\phi} + \theta sA\right)\left(\theta sA + \pi\widetilde{b}_{2t+1}\right)}, \tag{44}$$

where $\Psi(\widetilde{b}_{2t+1}) = 1$ when $\pi = 1$ and $\Psi(\widetilde{b}_{2t+1}) > 1$ when $\pi < 1$, meaning that $\Psi(\widetilde{b}_{2t+1})$ is decreasing in π.

As it is derived in Appendix A.2, an equilibrium is a sequence $\{\widetilde{b}_{2t}, g_t\}_{t\geqslant 0}$ satisfying the two following equations:

$$\widetilde{b}_{2t+1} = \frac{\alpha\frac{\phi(1-s)A}{1+\phi} + \theta sA\big[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\big]}{(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{2t}\big[\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})\big]}\widetilde{b}_{2t}, \tag{45}$$

$$g_{t+1} = \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi(1-s)}{1+\phi} + \theta s\right] + \frac{\theta s(1-s)A(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})}{\alpha\phi(1-s) + \theta s(1+\phi)[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1})]}$$

$$+\widetilde{b}_{2t}\left[\frac{\gamma}{\beta+\gamma}-\frac{\theta s}{\pi}\frac{\alpha+(\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})}{\frac{\alpha\phi(1-s)}{1+\phi}+\theta s[\alpha+(\beta+\gamma)\Psi(\widetilde{b}_{2t+1}))]}\right]. \tag{46}$$

1.4 Comparison with the Related Literature

Before analyzing equilibria, we more specifically compare our models with the closest related literature, namely, Farhi and Tirole (2012), Hirano and Yanagawa (2017), Martin and Ventura (2012), ,2016), Kocherlakota (2009), and Miao and Wang (2018).

Farhi and Tirole (2012) is a special case of the model with $b_{1t} > 0$ and $b_{2t} = 0$ for all t. We generalize their approach considering a general equilibrium model, meaning that prices and incomes are endogenous. Furthermore, we do not have to introduce an adding asset representing outside liquidity, called trees, which is required for their result.

Hirano and Yanagawa (2017) is quite similar to our framework when $b_{1t} > 0$ and $b_{2t} = 0$ for all t. Despite the fact that they consider infinitely lived agents, they distinguish between high and low productive investors. In our model, adult individuals correspond to high productive investors and young individuals to low productive investors, taking the extreme assumption that young individuals are completely unproductive. Having $b_{1t} > 0$ and $b_{2t} = 0$ in our framework means that the bubble is used only to transfer resources from less to more productive agents, as in their paper.

The mechanism for the crowding-in effect of the bubble highlighted by Martin and Ventura (2012) is also encompassed in our framework. They consider two-period lived overlapping generations in which agents are heterogeneous because their investments have different returns. Despite the fact that they have no credit, the bubble enhances growth because it reallocates resources from less to more productive traders, as in our framework with $b_{1t} > 0$ and $b_{2t} = 0$. In our model, it corresponds to a situation where the unproductive young agents buy the bubble from the productive adult ones. Note that in contrast to Martin and Ventura (2012), we will not need any exogenous bubble shocks to have a crowding-in effect of the bubble. Indeed, in their model, some new bubbles are distributed to young agents at each period. These new bubbles are those sold by the more productive agents to the less productive one generating the liquidity effect of the bubble. This also means that without these new bubbles, a crowding-in effect cannot occur and dominate the crowding-out effect on capital.

Our model also generalizes Martin and Ventura (2016) when $b_{1t} = 0$ and $b_{2t} > 0$ for all t. Indeed, we also have workers that provide credits (young agents in our framework) to some investors (adults in our framework), but this heterogeneity among agents comes from the three-period lifetime. In their model, the credit constraint also has two types of collateral, one related to the value of the firm and one associated to the bubble. However, note that in contrast to Martin and Ventura (2016), bubbles have a crowding-in effect without adding bubble shocks in our paper. If in Martin and Ventura (2016) there was no bubble shock, that is, new bubbles that enter the collateral constraint, there is no reason why credit constrained entrepreneurs will invest more. Indeed, otherwise, holding the bubble has a crowding-out effect on saving through capital.

The credit constraint investigated by Kocherlakota (2009) can also be seen as a particular case of our model if we set $\theta = 0$. This case where $\theta = 0$ corresponds to a configuration with strong financial imperfections in our framework, since capital does no more play the role of collateral.

Finally, Miao and Wang (2018) consider a model with heterogenous entrepreneurs facing idiosyncratic shocks. In their model, there is a bubble on the value of the firm, which is used as collateral in a credit constraint. Accordingly, the bubble has a collateral role as the one we investigate. In an extension of the initial model to take into account a pure bubble, Miao and Wang (2018) assume that this pure bubble can be sold by investing entrepreneurs to noninvesting ones to increase their investment. This mechanism is related to the liquidity effect we consider in the YS model.

2. DETERMINISTIC EQUILIBRIA

We first compare the equilibria of the models YS and AI when the bubble is deterministic. Then, we analyze the existence of BGPs (Balanced Growth Path) and the dynamics of the bubble. We study in particular whether the bubble enhances growth or not. We end this section giving an intuition for the crowding-in effect of the bubble.

2.1 Comparison of Liquidity and Collateral Roles of Bubbles

The bubble is deterministic when the probability of market crash is zero, which means that $\pi = 1$. In this case, the dynamic system (22)–(23), which defined an equilibrium in the model YS writes:

$$g_{t+1} = \frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)} + \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right]$$

$$+b_t\left[\frac{\gamma}{\beta+\gamma} - \theta s\frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right] \equiv F(b_t), \tag{47}$$

$$b_{t+1} = \theta s A\frac{1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t}b_t \equiv G(b_t) \tag{48}$$

with $b_t = \widetilde{b}_{1t}$.

Substituting now $\pi = 1$ in the dynamic system (45)–(46) which defines an intertemporal equilibrium in the model AI, we obtain the dynamic system (47)–(48) with $b_t = \widetilde{b}_{2t}$.

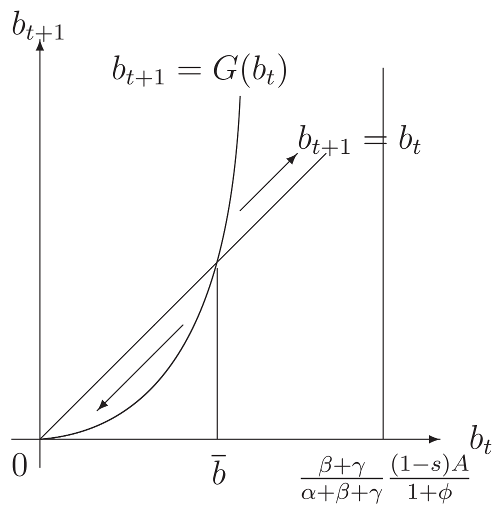

Fig 3. Dynamics of Bubble.

The equilibrium of the models YS and AI is characterized by the same equations, which means that they share the same equilibrium. There is a perfect equivalence between the models YS and AI, despite the fact that the role of the bubble is *a priori* different. In the model YS, the bubble is introduced to provide liquidity to the adult investors. In contrast, in the model AI, the bubble is bought by adult investors that use it as a collateral to borrow. Therefore, the bubble increases the amount borrowed by the adult individuals and, as a result, the deposits of the young individuals also increase.

To gain some intuition on this equivalence result, we next show that the effect of the bubble on the savings of the young coincides in the two models. To see this, we use the binding credit constraint to show that, in the AI model, these savings are given by $d_{1t} = \theta q_{t+1}k_{t+1}/R_{t+1} + b_{2t}$, whereas in the YS model, they are given by $d_{1t} + b_{1t} = \theta q_{t+1}k_{t+1}/R^d_{t+1} + b_{1t}$. It explains that the two mechanisms play exactly the same role and the bubble has finally the same effect in both models, as follows from the fact that the reduced forms of the two models are identical.

2.2 BGPs, Dynamics, and Crowding-In Effect of the Bubble

We study now the dynamic system (47)–(48), which describes the equilibrium of both the AI and YS models, to examine the existence of bubbleless and bubbly BGPs, the dynamics and whether the existence of the bubble is associated to more growth, that is, has a crowding-in effect (see Figure 3).

For further reference, equation (48) can be written:

$$g_{t+1}b_{t+1} = R_{t+1}b_{t}, \tag{49}$$

where the interest factor is given by

$$R_{t+1} = \theta s A g_{t+1} \frac{1 + \frac{\alpha}{\alpha+\beta+\gamma} \frac{\phi(1-s)}{\theta s(1+\phi)}}{\frac{\beta+\gamma}{\alpha+\beta+\gamma} \frac{(1-s)A}{1+\phi} - b_t} \equiv R(g_{t+1}, b_t) \tag{50}$$

with $b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma} \frac{(1-s)A}{1+\phi}$. Equation (49) gives the dynamics of the ratio of bubble over capital, which is a nonpredetermined variable. Given the sequence of $\{b_t\}_{t\geqslant 0}$, we deduce the growth factor at each period of time using (47).

There exist two BGPs, the bubbleless one $(\underline{b}, \underline{g}) = (0, F(0))$ and the bubbly one $(\overline{b}, \overline{g}) = (\overline{b}, F(\overline{b}))$, with:

$$\overline{b} \equiv \frac{\beta + \gamma - \alpha\phi}{\alpha + \beta + \gamma} \frac{(1 - s)A}{1 + \phi} - \theta s A, \tag{51}$$

where $\overline{b} > 0$ if:

$$\phi < \frac{\beta + \gamma}{\alpha} \quad \text{and} \quad \theta < \frac{\beta + \gamma - \alpha\phi}{\alpha + \beta + \gamma} \frac{(1 - s)}{s(1 + \phi)} \equiv \theta_a \tag{52}$$

and

$$\overline{g} = A\left[s\theta + (1 - s)\frac{\gamma}{\alpha + \beta + \gamma}\right]. \tag{53}$$

Taking into account conditions (52), we can easily prove that the bubbleless steady state $\underline{b}$ is stable and the bubbly steady state $\overline{b}$ is unstable. Therefore, there are three types of equilibria depending on agents' expectations:

- there is no bubble, $b_t = \underline{b} = 0$;
- there is a persistent bubble, $b_t = \overline{b} > 0$;
- there is a bubble that decreases and converges to 0 for all $0 < b_t < \overline{b}$.

If individuals initially choose b_t such that $\overline{b} < b_t < \frac{\beta+\gamma}{\alpha+\beta+\gamma} \frac{(1-s)A}{1+\phi}$, the bubble would increase along this equilibrium, and it would eventually crash after a finite number of periods when b_t crosses the upper bound $\frac{\beta+\gamma}{\alpha+\beta+\gamma} \frac{(1-s)A}{1+\phi}$. Therefore, rational individuals will never buy the bubble at such a price, and hence, this is never an equilibrium. All equilibria must satisfy $0 \leqslant b_t \leqslant \overline{b}$.[13]

Of course, these equilibria should satisfy the binding borrowing constraint, that is, $q_{t+1} > R_{t+1} > \theta q_{t+1}$.

13. Note that if the bubbly BGP $\overline{b}$ does not exist, that is, $\phi > \frac{\beta+\gamma}{\alpha}$ or $\theta > \theta_a$, we can easily show, using the same arguments than above, that the only equilibrium is the bubbleless BGP.

LEMMA 1. *Any equilibrium $b_t \in [0, \overline{b}]$ satisfies the binding borrowing constraint if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$, with:*

$$\theta_b \equiv \alpha\phi \frac{1 - \frac{1-s}{s}\frac{\gamma}{\alpha+\beta+\gamma}}{\alpha\phi + \gamma(1+\phi)}. \tag{54}$$

PROOF. See Appendix A.3. □

Using this lemma, we deduce the existence of the different types of equilibria with bubble:

PROPOSITION 1. *Assuming $\phi < \frac{\beta+\gamma}{\alpha}$, $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$, and $\theta < \min\{\theta_a, \theta_b\}$, there exist three types of equilibria:*

(i) a bubbleless BGP $b = \underline{b} = 0$;
(ii) a bubbly BGP $b = \overline{b}$;
(iii) any sequence $b_t \in (0, \overline{b})$, which decreases and converges to 0.

The bubble has a crowding-in effect on growth if and only if there is a positive relationship between b_t and g_{t+1}. The following proposition summarizes the main results:

PROPOSITION 2. *Let*

$$\widehat{\theta} \equiv \frac{\gamma}{\beta}\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{s(1+\phi)}. \tag{55}$$

(i) If we have $\phi < \frac{\beta}{\alpha}$ and $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$, the bubble has a crowding-in effect on growth if $\theta < \widehat{\theta}$, has no effect on growth if $\theta = \widehat{\theta}$ and has a crowding-out effect on growth if $\widehat{\theta} < \theta < \min\{\theta_a, \theta_b\}$.[14]
(ii) If either $\frac{\beta}{\alpha} \leqslant \phi < \frac{\beta+\gamma}{\alpha}$ or $\frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} \geqslant \frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$, the bubble has a crowding-in effect on growth for all $\theta < \min\{\theta_a, \theta_b\}$.

PROOF. See Appendix A.4. □

This proposition shows that when $\theta < \widehat{\theta}$, the bubble boosts growth. In this case, the crowding-in effect of the bubble dominates its crowding-out effect. As a direct implication, the bubbly BGP features a higher growth than the bubbleless one, that is, $\overline{g} > \underline{g}$.[15] In contrast, when $\theta > \widehat{\theta}$, the crowding-out effect of the bubble dominates its crowding-in effect and $\overline{g} < \underline{g}$. A lower θ, that is, a stronger credit market imperfection, reinforces the crowding-in effect of the bubble.

This result can be related to Hirano and Yanagawa (2017) who also analyzes an endogenous growth model, but with heterogeneous infinitely lived agents. In their

14. Note that $\phi < \frac{\beta}{\alpha}$ and $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma}$ ensure that $\widehat{\theta} < \min\{\theta_a, \theta_b\}$.

15. We also observe that the interest factor at the bubbleless BGP $\underline{R}$ is lower than at the bubbly BGP, $\overline{R}$.

framework, there is also a level of θ such that below it, the crowding-in effect dominates, whereas above it, the bubble has a crowding-out effect. However, in contrast to us, the existence of the bubble requires a minimum value for θ. The main difference lies in the returns of investment opportunities. In their framework, all investment opportunities have a positive return, whereas in our model, young agents do not invest in capital because they expect a zero return. More importantly, our results hold whatever the type of bubble considered, that is, either young agents buy the bubble ($b_{1t} > 0$), or adults ($b_{2t} > 0$).

Finally, in addition of having a crowding-in effect, we show now that the bubble is welfare improving:

PROPOSITION 3. *Assuming* $\phi < \frac{\beta+\gamma}{\alpha}$, $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$, *and* θ *low enough, a bubbly equilibrium such that* $b_t \in (0, \overline{b}]$ *is characterized by a higher welfare for all generations than a bubbleless BGP with* $\underline{b} = 0$.

PROOF. See Appendix A.5. □

Welfare increases with the life-cycle labor income and decreases with the interest rate. Since the bubble increases the interest rate, it can only increase welfare when there is a large crowding-in effect that increases life-cycle labor income. Note that this contrasts with the result obtained by Hirano and Yanagawa (2017) where the bubble always improves welfare because of a better consumption smoothing. In our case, the important channel goes through the effect of growth on wage, and therefore life-cycle labor income.

2.3 *Economic Interpretation*

We would like to understand why the bubble can promote growth when the adults/investors are constrained. Note that if there was no binding borrowing constraint, households could perfectly smooth consumption and all assets would be perfect substitutes, that is, would have the same return. In this case, consumptions would linearly depend on the life-cycle income, which would imply that total savings would not depend on the bubble. As a direct implication, any increase of the bubble would imply a decrease of the new investment in capital.

When adults face a binding borrowing constraint, households cannot smooth consumption without any restrictions, and capital is no more substitutable for credit and bubble. The consumptions depend now on the asset holdings and, therefore, the savings too. This opens the door to mechanisms for which the bubble has a crowding-in effect on capital.

Since $c_{3t+2} = (1-\theta)q_{t+2}k_{t+2}$, we first observe that the consumption when old does not directly depend on the level of the bubble because of the binding borrowing constraint. The redistribution from the old to the adult age coming from borrowing only depends on capital income, because the loan demand net of the purchase or sale of the bubble is constrained by the fundamental collateral. Focusing now on consumptions when young and adult, we easily see that the bubble induces a redistribution from the young households to the adults. In other words, it increases savings, as we have already shown. If the bubble is bought when young ($b_{1t} > 0$), we have the liquidity effect, which may cause a crowding-in effect of the bubble. If the bubble is bought when adult ($b_{2t} > 0$), the explanation goes through the credit market. Due to the binding borrowing constraint, a higher bubble leads to more loans. To satisfy the equilibrium on the credit market, these loans are financed by deposits of young households. Therefore, a higher bubble requires more deposits by young households. Since these deposits are remunerated at the next period, this increases the liquidity transferred at the adult age. Since deposits, or credit, and bubbles are perfectly substitutable assets, the liquidity role of both types of bubbles (b_{1t} or b_{2t}) is identical.

Finally, the existence of a bubble, which means a higher supply of liquid assets than at the bubbleless equilibrium, also increases the cost of credit used to finance capital (R_{t+1} increases), which reduces capital investment (see, for instance, (43) with $\pi = 1$). Therefore, the bubble enhances growth when the degree of pledgeability θ is sufficiently small. The higher θ, the more important the negative effect of a raise in R_{t+1} on capital. This last effect corresponds to the crowding-out effect of the bubble when the borrowing constraint is binding.

3. STOCHASTIC EQUILIBRIA

When the model is deterministic, a bubble is a perfect substitute of credit. The liquidity and collateral roles of the bubble are equivalent despite the fact that the first role is obtained when the bubble is bought when young and the second one when the bubble is bought at the middle age. When we enrich the model with a positive probability of bubble crash, bubbles and credit are no more perfect substitutes and the difference in the period bubbles are bought makes the two roles of the bubble different. In what follows, we compare the bubble size and growth rate generated by these two roles attributed to the bubble, when this latter is stochastic. This comparison is relevant for several reasons, as for instance to know whether a household invests more when the bubble is bought at young or adult age. It is also relevant for governments that might like to know in which case the crash of the bubble will be more damaging, at least to facilitate the implementation of the most appropriate policy.

3.1 Model YS with a Stochastic Bubble

As we have seen in Section 2.2, an equilibrium with stochastic bubble in the model YS is a sequence $\{\widetilde{b}_{1t}, g_t\}_{t \geqslant 0}$ satisfying equations (22) and (23). The dynamics are qualitatively similar than in the deterministic model (see Section 3) for all $\widetilde{b}_{1t} < \widehat{b}_1$, with $\widehat{b}_1 \equiv \frac{\pi(1-s)A}{1+\phi} / [\frac{\alpha+\pi(\beta+\gamma)}{\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}]$.

If we further compare with the deterministic case (see equation (48)), we note that for a given $\widetilde{b}_{1t}$, $I(\widetilde{b}_{1t})$ decreases with respect to π. Since $I(\widetilde{b}_{1t})$ is an increasing and convex function, it means that the higher the probability of bubble crash $1 - \pi$, the smaller is the level of the bubble per unit of capital $\widetilde{b}_1$ at the stochastic bubbly BGP, with:

$$\widetilde{b}_1 = \frac{\frac{(1-s)A}{1+\phi}\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\beta+\gamma} - \theta sA}{\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}}, \tag{56}$$

which is strictly positive for $\pi\frac{\beta+\gamma}{\alpha} > \phi$ and $\theta < \frac{(1-s)[\pi(\beta+\gamma)-\alpha\phi]}{s(\alpha+\beta+\gamma)(1+\phi)}$. In addition, the higher the probability $1-\pi$, the smaller the range of $\widetilde{b}_{1t} \in (0, \widetilde{b}_1)$ converging to the bubbleless BGP.

Once the equilibrium path of the bubble is determined, equation (22) gives the equilibrium level of growth. Equation (22) is identical to (47), which determines the growth rate as a function of the bubble in the deterministic model. We have seen that, if θ is sufficiently low, there is a positive relationship between the growth rate and the bubble size. In this case, the growth rate at the stochastic bubbly BGP is higher than at the bubbleless BGP. This means that the bubble has a crowding-in effect.

Because of the positive probability of market crash, it is risky for a young household to buy the bubble. Therefore, the bubble size is lower than at a deterministic bubbly BGP. As a direct implication, the growth rate is also lower than in the deterministic case when the bubble is productive. Although adult investors do not face any risk associated to bubble burst, they invest less in capital because they own less liquidities than under a deterministic bubble.

3.2 *Model AI with a Stochastic Bubble*

In Section 2.3, we have shown that an equilibrium with stochastic bubble in the model AI is a sequence $\{\widetilde{b}_{2t}, g_t\}_{t\geqslant 0}$ satisfying equations (45) and (46). To keep things as simple as possible, we focus on the existence and properties of BGPs. By inspection of equation (45), we see that a bubbleless BGP $\widetilde{b}_2 = 0$ exists, while a stochastic bubbly BGP with a positive bubble can be defined by:

$$\widetilde{b}_2 = \frac{\frac{(1-s)A}{1+\phi}\left[(\beta+\gamma)\pi\Psi(\widetilde{b}_2) - \alpha\phi\right] - \theta sA\left[\alpha + (\beta+\gamma)\Psi(\widetilde{b}_2)\right]}{\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_2)} \equiv B(\Psi(\widetilde{b}_2)) \tag{57}$$

In Appendix A.6, we show that a unique positive solution $\widetilde{b}_2 > 0$ exists if $\phi < \pi\frac{\beta+\gamma}{\alpha}$ and $\theta < \min\{\frac{\pi(1-s)}{(1-\pi)s}; \frac{1-s}{s(1+\phi)}\frac{(\beta+\gamma)\pi-\alpha\phi}{\alpha+\beta+\gamma}\}$.

To compare with the deterministic bubbly BGP, we note that $\pi\Psi(\widetilde{b}_2) < 1$ and $\Psi(\widetilde{b}_2) > 1$ when $\pi < 1$, whereas they are equal to 1 when $\pi = 1$. From equations (51) and (57), we deduce that the stochastic bubble $\widetilde{b}_2$ at the BGP is lower than in the deterministic case. When there is a positive probability of market crash, the bubble size is lower than when the bubble is riskless. This result is similar to that obtained when the bubble has a liquidity role, but the interpretation is different. Now, the bubble does not increase risk, since the household faces a form of consumption insurance at the old age. However, the loans collateralized by capital increase relative to the loans collateralized by the bubble, because the return of capital becomes relatively higher (see equation (39)). This explains the lower value of the bubble when $\pi < 1$. Finally, similarly to the model YS, equation (46) allows us to deduce that for θ low enough, the bubble has a positive effect on growth, meaning that there is still a crowding-in effect of the bubble.

We underline at this stage that we assume that the loans d_2 ensure perfect consumption insurance to match the deterministic model in the limit case with $\pi = 1$. Other properties could characterize loans. For instance, if we consider that loans are remunerated by the same interest rate whatever the state of nature, the borrowing constraint would no more be binding at each state of nature in the AI model. This might rule out the crowding-in effect of the bubble. Our specification has been chosen so that the credit constraint binds in every state, which is in accordance with existing models dealing with a bubbly collateral (Kocherlakota 2009, Martin and Ventura 2016, Miao and Wang 2018).

3.3 Comparison of the Liquidity and Collateral Roles of a Stochastic Bubble

To compare the liquidity and the collateral roles of a stochastic bubble, we first study if $\widetilde{b}_1$, given by (56), is higher or not than $\widetilde{b}_2$, defined by (57). As it is shown in Appendix A.6, we have $B'(\Psi) > 0$, which means that $B(\Psi(\widetilde{b}_2)) \geq B(1)$ because $\Psi(\widetilde{b}_2) \geq 1$. Using (56) and (57), we easily deduce that $\widetilde{b}_2 \geq B(1) \geq \widetilde{b}_1$.

Second, we recall that we measure the importance of the crowding-in effect as the difference between the growth rate at the bubbly and at the bubbleless BGPs, where the last one is of course the same whatever the role played by the bubble. Therefore, to determine whether the crowding-in effect is more important when the bubble has a liquidity or a collateral role, we compare the growth rates at the stochastic bubbly BGPs in the YS and AI models.

Let us call g_1 the growth factor at the stochastic bubbly BGP in the model YS and g_2 the growth factor at the stochastic bubbly BGP in the model AI. The first one identifies the liquidity effect when the bubble is stochastic, the second one the collateral effect. We show the following results:

PROPOSITION 4. *Assuming that $\pi < 1$ and $\pi(\beta + \gamma) > \alpha\phi$, we have:*

(i) $\widetilde{b}_1 = \widetilde{b}_2$ and $g_1 = g_2$ if $\theta = 0$;
(ii) $\widetilde{b}_1 < \widetilde{b}_2$ and $g_1 > g_2$ if $\theta > 0$ is low enough.

PROOF. See Appendix A.7. □

When there is no fundamental collateral, $\theta = 0$, we recover the result that we obtain when the bubble is deterministic. The liquidity and collateral roles of the bubble are identical since the stochastic bubbly BGP is characterized by the same levels of bubble and growth.

With a fundamental collateral in the credit constraint, that is, $\theta > 0$, the collateral role of the bubble generates a higher bubble than the liquidity role. However, the crowding-in effect is stronger under the liquidity role than under the collateral role.

When $\pi < 1$, the stochastic bubbles in both models become lower than the deterministic one. At a stochastic BGP, the bubble is higher when it plays the role of collateral than the role of liquidity, because this asset is riskier in terms of consumption in the model YS than in the model AI. Indeed, when a young household buys b_1, he faces the risk of a bubble crash at the adult age. In addition, savings through deposits when young is less risky since it gives the same return whatever the value of the bubble. A larger share of saving at young age is therefore composed of deposits. On the contrary, when an adult buys b_2 in the model AI, he faces no risk in terms of future consumption, since the loans d_2 provide a consumption insurance against the risk of a bubble crash in the last period of life and the expected return of the bubble is equal to the expected return of loans (see equation (38)).

We explain now why growth, and therefore, investment in capital are larger in the model YS than in the model AI. By inspection of (17) and (18), we observe that, in the model YS, the cost of borrowing R^d_{t+1} is smaller than the return of the bubble. In contrast, in the model AI, the cost of borrowing is larger since it is equal to the expected return of the bubble (see (38)). As a result, capital investment is higher in the model YS than in the model AI, because the investment multiplier and deposits used to finance loans collateralized by capital are higher. This explains that the liquidity and collateral roles are no more completely equivalent when the bubble is stochastic. However, despite their quantitative difference when $\theta > 0$, the collateral and liquidity roles have still comparable effects on growth, since with both of them, the stochastic bubble has a crowding-in effect on growth.

This last conclusion is reinforced by the limit case where $\theta = 0$, where the bubble and growth are the same in both models. As there is no fundamental collateral, the cost of borrowing has no impact on capital investment. The investment multiplier is equal to one in both models and loans collateralized by capital are null. According to our previous explanation, there is no difference in capital investment. Moreover, in the model YS, deposits become zero and young agents save only through the bubble, while in the model AI, deposits finance loans which correspond to the bubbly collateral. Therefore, in both models, young agents save using the same asset, the bubble, which then has the same value. The same type of intuition explains that the differences between the two models remain small when θ is small.

4. CONCLUDING REMARKS

Recently, several papers have identified some channels through which asset bubbles promote economic activity. Two important common features in these papers are the existence of borrowing constraints and the heterogeneity of traders. However, bubbles have different roles. The two main ones are to provide liquidities and to serve as a collateral. In this paper, we introduce heterogeneous traders by considering an overlapping generations model with three period-lived households. Only adults have access to capital investment, and face a borrowing constraint. We show that the roles played by a deterministic bubble, namely, to provide liquidities and to be a collateral, are perfectly equivalent. Therefore, the equilibrium is identical under these two roles of the bubble. We show that bubbles may increase growth and also welfare.

When there is a stochastic bubble, which may crash at each period of time with a positive probability, the equilibrium with the liquidity and collateral roles of the bubble is not identical. However, both equilibria are still very similar even if some differences appear when capital is used as collateral. Because the liquidity role is more damaging in terms of consumption when a bubble crashes, the bubble size is lower than when the bubble has a collateral role. On the contrary, the liquidity role generates higher growth, because the burst of a bubble used for liquidity is less damaging to agents who invest in capital.

As we have seen, the asset bubble and the credit market allow for some transfers from the young and old agents to adults who invest in capital. Of course, if the young agents invested in capital instead of the adults, the conclusions would be completely different. As shown by Raurich and Seegmuller (2019), the transfers would be done from the adult age to the young and old ones. In their paper, the bubble enhances production because its existence relaxes the binding credit constraint and facilitates investment. Our paper is complementary to this previous one. Using these two contributions, the conditions for the existence of a crowding-in effect of the bubble are established, regardless the investment in capital is done at the young or at the adult age.

APPENDIX A

A.1 Derivation of an Equilibrium in the Model YS (Young Savers) with a Stochastic Bubble

Using (13), (15), and (20), we deduce that:

$$c^0_{2t+1} = \frac{\beta}{\beta+\gamma}(\phi w_{t+1} + R^d_{t+1} d_{1t}), \tag{A.1}$$

and using (12), (14), and (19),

$$c^+_{2t+1} = \frac{\beta}{\beta+\gamma}(\phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t}), \tag{A.2}$$

$$k_{t+2} = \frac{\gamma}{\beta+\gamma}\frac{\phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t}}{1-\theta\frac{q_{t+2}}{R^d_{t+2}}}. \tag{A.3}$$

We use now (17) and (18), and substitute the consumptions (A.1) and (A.2) to get the arbitrage condition:

$$b_{1t+1} = \frac{\pi r_{t+1}/R^d_{t+1} - 1}{1-\pi}(\phi w_{t+1} + R^d_{t+1} d_{1t}), \tag{A.4}$$

which shows that the return of the bubble r_{t+1} times the probability of bubble persistence π should be higher than the return on deposits R^d_{t+1} for there to be a bubble.

Using the equilibrium prices (3) and (4), the equilibrium on the debt market $d_{1t} = -d_{2t} = \theta q_{t+1}k_{t+1}/R^d_{t+1}$, and the variables $k_t = (1+\phi)a_t$, $g_{t+1} = a_{t+1}/a_t$, and $\widetilde{b}_{1t} = b_{1t}/[(1+\phi)a_t]$, equation (A.3) implies that:

$$g_{t+1} = \frac{\gamma}{\beta+\gamma}\frac{\frac{\phi}{1+\phi}(1-s)A + \theta sA + \widetilde{b}_{1t}}{1-\theta\frac{sA}{R^d_{t+1}}}. \tag{A.5}$$

Substituting the budget constraint (7) and the consumption (A.2) in the first-order condition (18), we get:

$$\left[\frac{\alpha}{\beta+\gamma}\frac{\phi}{1+\phi}(1-s)A + \theta sA\left(\frac{\alpha}{\beta+\gamma} + \pi\frac{r_{t+1}}{R^d_{t+1}}\right)\right]g_{t+1}$$
$$= r_{t+1}\left[\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left(\pi + \frac{\alpha}{\beta+\gamma}\right)\right], \tag{A.6}$$

while the arbitrage condition (A.4) writes:

$$\frac{(1-\pi)(1+\phi)}{\phi(1-s)A + \theta sA(1+\phi)}\widetilde{b}_{1t+1} = \pi r_{t+1}/R^d_{t+1} - 1 \tag{A.7}$$

and the evolution of the bubble $b_{1t+1} = r_{t+1}b_{1t}$ is equivalent to:

$$g_{t+1}\widetilde{b}_{1t+1} = r_{t+1}\widetilde{b}_{1t}. \tag{A.8}$$

Now, we can use (A.7) to substitute R^d_{t+1} in (A.6), and deduce the return of the bubble:

$$r_{t+1} = \frac{g_{t+1}\left[\frac{\alpha}{\beta+\gamma}\frac{\phi}{1+\phi}(1-s)A + \theta sA\frac{\alpha+\beta+\gamma}{\beta+\gamma}\right]}{\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{1t}\left[\pi + \frac{\alpha}{\beta+\gamma} + \theta s\frac{(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\right]}. \tag{A.9}$$

Using (A.7) and (A.9), equation (A.5) gives the growth factor as a function of the level of the bubble. Finally, substituting (A.9) in (A.8), we deduce the dynamic path of the bubble. The resulting equations define the dynamic system (22)–(23).

A.2 Derivation of an Equilibrium in the Model AI (Adult Investors) with a Stochastic Bubble

Substituting (24), (34), and (40) in the arbitrage condition (33), we get:

$$\frac{\alpha}{w_t - d_{1t}} = \frac{(\beta+\gamma)\pi R^d_{t+1}}{\phi w_{t+1} + R^d_{t+1}d_{1t}} + \frac{(\beta+\gamma)(1-\pi)\widetilde{R}^d_{t+1}}{\phi w_{t+1} + \widetilde{R}^d_{t+1}d_{1t}}. \tag{A.10}$$

Using (38) and (39), and the equilibrium conditions which follows from these equations, we can substitute R^d_{t+1}, d_{1t}, and $\widetilde{R}^d_{t+1}$ to obtain:

$$\frac{\alpha(\theta q_{t+1}k_{t+1} + \pi R_{t+1}b_{2t})}{\pi R_{t+1}(w_t - b_{2t}) - \theta q_{t+1}k_{t+1}} = \frac{(\beta+\gamma)\pi(\theta q_{t+1}k_{t+1} + R_{t+1}b_{2t})}{\phi w_{t+1} + \theta q_{t+1}k_{t+1} + R_{t+1}b_{2t}} + \frac{(\beta+\gamma)(1-\pi)\theta q_{t+1}k_{t+1}}{\phi w_{t+1} + \theta q_{t+1}k_{t+1}}. \quad \text{(A.11)}$$

Recall that, when the bubble exists, b_{2t} is its price at time t and R_{t+1} is the increase of this price. Accordingly, the evolution of the bubble $b_{2t+1} = R_{t+1}b_{2t}$ writes:

$$g_{t+1}\widetilde{b}_{2t+1} = R_{t+1}\widetilde{b}_{2t} \quad \text{(A.12)}$$

and, using (3) and (4), equation (43) is equivalent to:

$$g_{t+1}\left(1 - \theta\frac{sA}{\pi R_{t+1}}\right) = \frac{\gamma}{\beta+\gamma}\left[\frac{\phi(1-s)A}{1+\phi} + \widetilde{b}_{2t} + \theta sA\right]. \quad \text{(A.13)}$$

Using (3)–(4) again and (A.11), we obtain:

$$\frac{\alpha g_{t+1}\left(\frac{\phi(1-s)A}{1+\phi} + \theta sA\right) + \alpha R_{t+1}\widetilde{b}_{2t}}{\pi R_{t+1}\left(\frac{(1-s)A}{1+\phi} - \widetilde{b}_{2t}\right) - \theta sAg_{t+1}} = (\beta+\gamma)\Psi(\widetilde{b}_{2t+1}), \quad \text{(A.14)}$$

where $\Psi(\widetilde{b}_{2t+1})$ is defined by (44). Note that (A.14) is equivalent to:

$$g_{t+1}\left[\alpha\frac{\phi(1-s)A}{1+\phi} + \theta sA\big(\alpha + (\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\big)\right] \quad \text{(A.15)}$$
$$= R_{t+1}\left[(\beta+\gamma)\Psi(\widetilde{b}_{2t+1})\pi\frac{(1-s)A}{1+\phi} - \widetilde{b}_{2t}\big(\alpha + (\beta+\gamma)\pi\Psi(\widetilde{b}_{2t+1})\big)\right].$$

Substituting R_{t+1}/g_{t+1} in the evolution of the bubble (A.12) and in (A.13), we get (45) and (46).

A.3 Proof of Lemma 1

Using (2) and (50), $R_{t+1} > \theta q_{t+1}$ is equivalent to:

$$g_{t+1}\left[1 + \frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right] > \frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi} - b_t. \quad \text{(A.16)}$$

By inspection of (47), this inequality is always satisfied. Using again (2) and (50), $R_{t+1} < q_{t+1}$ is equivalent to:

$$RHS(b_t) \equiv \frac{\theta\gamma}{\beta+\gamma}\left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\left[b_t+\frac{\phi(1-s)A}{1+\phi}+\theta sA\right]$$
$$<(1-\theta)\left[\frac{\beta+\gamma}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}-b_t\right] \equiv LHS(b_t). \tag{A.17}$$

For $b_t \leqslant \overline{b}$, we deduce that:

$$RHS(b_t) \leqslant \left[1+\frac{\alpha}{\alpha+\beta+\gamma}\frac{\phi(1-s)}{\theta s(1+\phi)}\right]\frac{(1-s)A\theta\gamma}{\alpha+\beta+\gamma},$$
$$LHS(b_t) > (1-\theta)\frac{\alpha\phi}{\alpha+\beta+\gamma}\frac{(1-s)A}{1+\phi}.$$

Using these last two inequalities, inequality (A.17) is satisfied if $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$ and $\theta < \theta_b$.

A.4 Proof of Proposition 2

Assume $\phi < \frac{\beta+\gamma}{\alpha}$, $\frac{s}{1-s} > \frac{\gamma}{\alpha+\beta+\gamma}$, and $\theta < \min\{\theta_a, \theta_b\}$. Using (47), g_{t+1} is increasing (decreasing) in b_t if and only if $\theta < \widehat{\theta}$ $(\theta > \widehat{\theta})$ and g_{t+1} does not depend on b_t if and only if $\theta = \widehat{\theta}$. Then, using (52) and (55), we can show that $\widehat{\theta} < \theta_a$ is equivalent to:

$$\phi < \frac{\beta}{\alpha} < \frac{\beta+\gamma}{\alpha}.$$

Using now (54) and (55), $\widehat{\theta} < \theta_b$ if and only if:

$$\phi[\gamma(1-s)-\beta s] < \beta s-(1-s)\gamma\frac{\beta+\gamma}{\alpha+\beta+\gamma},$$

which is equivalent to $\frac{s}{1-s} > \frac{\gamma}{\beta(1+\phi)}\frac{\alpha\phi+(\beta+\gamma)(1+\phi)}{\alpha+\beta+\gamma} > \frac{\gamma}{\alpha+\beta+\gamma}$. Using Proposition 1, we easily deduce the proposition.

A.5 Proof of Proposition 3

Since the collateral and liquidity roles of the bubble are similar in the deterministic case, we consider the model YS to fix ideas. Taking into account that the borrowing constraint is binding, that is, $-R^d_{t+2}d_{2t+1} = \theta q_{t+2}k_{t+2}$, the budget constraints write:

$$c_{1t} + d_{1t} + b_{1t} = w_t, \tag{A.18}$$

$$c_{2t+1} + k_{t+2}\left(1 - \theta \frac{q_{t+2}}{R^d_{t+2}}\right) = \phi w_{t+1} + R^d_{t+1} d_{1t} + r_{t+1} b_{1t,} \tag{A.19}$$

$$c_{3t+2} = (1-\theta) q_{t+2} k_{t+2.} \tag{A.20}$$

Since d_{1t} and b_{1t} are perfectly substitutable assets, we have $R^d_{t+1} = r_{t+1}$. Therefore, we can deduce the life-cycle budget constraint:

$$r_{t+1} c_{1t} + c_{2t+1} + \frac{1 - \theta q_{t+2}/r_{t+2}}{(1-\theta) q_{t+2}} c_{3t+2} = \phi w_{t+1} + r_{t+1} w_t. \tag{A.21}$$

Maximizing the utility function:

$$U_t = \alpha \ln(c_{1t}) + \beta \ln(c_{2t+1}) + \gamma \ln(c_{3t+2}) \tag{A.22}$$

under the constraint (A.21), we easily get:

$$\frac{c_{1t}}{w_{t+1}} = \frac{\alpha}{r_{t+1}(\alpha+\beta+\gamma)}\left(\phi + r_{t+1}\frac{w_t}{w_{t+1}}\right), \tag{A.23}$$

$$\frac{c_{2t+1}}{w_{t+1}} = \frac{\beta}{\alpha+\beta+\gamma}\left(\phi + r_{t+1}\frac{w_t}{w_{t+1}}\right), \tag{A.24}$$

$$\frac{c_{3t+2}}{w_{t+1}} = \frac{\gamma}{\alpha+\beta+\gamma}\frac{(1-\theta)q_{t+2}}{1-\theta q_{t+2}/r_{t+2}}\left(\phi + r_{t+1}\frac{w_t}{w_{t+1}}\right). \tag{A.25}$$

Using (3), (4), and $g_{t+1} = a_{t+1}/a_t$, these three equations rewrite:

$$\frac{c_{1t}}{w_{t+1}} = \frac{\alpha}{r_{t+1}(\alpha+\beta+\gamma)}\left(\phi + \frac{r_{t+1}}{g_{t+1}}\right), \tag{A.26}$$

$$\frac{c_{2t+1}}{w_{t+1}} = \frac{\beta}{\alpha+\beta+\gamma}\left(\phi + \frac{r_{t+1}}{g_{t+1}}\right), \tag{A.27}$$

$$\frac{c_{3t+2}}{w_{t+1}} = \frac{\gamma}{\alpha+\beta+\gamma}\frac{(1-\theta)sA}{1-\theta q/r_{t+2}}\left(\phi + \frac{r_{t+1}}{g_{t+1}}\right), \tag{A.28}$$

with

$$w_{t+1} = (1-s)Aa_0 \prod_{h=1}^{t+1} g_h. \tag{A.29}$$

Substituting (A.26)–(A.29) in (A.22), we get:

$$U_t = U_0 + (\alpha+\beta+\gamma)\ln\prod_{h=1}^{t} g_h + (\beta+\gamma)\ln g_{t+1} - \alpha\ln\left(\frac{r_{t+1}}{g_{t+1}}\right)$$
$$-\gamma\ln\left(1-\theta\frac{sA}{r_{t+2}}\right) + (\alpha+\beta+\gamma)\ln\left(\phi+\frac{r_{t+1}}{g_{t+1}}\right) \tag{A.30}$$

with $U_0 \equiv (\alpha+\beta+\gamma)\ln(1-s)Aa_0 + \alpha\ln\alpha + \beta\ln\beta + \gamma\ln\gamma - (\alpha+\beta+\gamma)\ln(\alpha+\beta+\gamma) + \gamma\ln(1-\theta)sA$.

We can easily see that the expression $(\alpha+\beta+\gamma)\ln(\phi+\frac{r_{t+1}}{g_{t+1}}) - \alpha\ln(\frac{r_{t+1}}{g_{t+1}})$ is increasing in r_{t+1}/g_{t+1} because $r_{t+1}/g_{t+1} > \underline{r}/\underline{g} > \alpha\phi/(\beta+\gamma)$, where $\underline{r} = R(\underline{g}, 0)$ is given by (50).

Since the existence of a bubble increases g_h, r_{t+1}, and r_{t+1}/g_{t+1}, U_t increases if θ is low enough. Proposition 3 immediately follows.

A.6 Existence and Uniqueness of a Stochastic Bubbly BGP in the Model AI

Using (57), we can show that $B'(\Psi) > 0$ and $B''(\Psi) < 0$ for $\theta < \frac{\pi(1-s)}{(1-\pi)s}$. Moreover, $\Psi'(\widetilde{b}_2) > 0$ and $\Psi''(\widetilde{b}_2) < 0$, which implies that $B(\Psi(\widetilde{b}_2))$ is increasing and strictly concave in $\widetilde{b}_2$ because $B'(\Psi)\Psi'(\widetilde{b}_2) > 0$ and $B''(\Psi)\Psi'(\widetilde{b}_2)^2 + B'(\Psi)\Psi''(\widetilde{b}_2) < 0$.

Therefore, there is a unique positive solution $\widetilde{b}_2 > 0$ to equation (57) if $B(\Psi(0)) > 0$. This is ensured by $\phi < \pi\frac{\beta+\gamma}{\alpha}$ and $\theta < \frac{1-s}{s(1+\phi)}\frac{(\beta+\gamma)\pi-\alpha\phi}{\alpha+\beta+\gamma}$.

A.7 Comparison of the Stochastic Bubbly Growth Rates in the YS and AI Models

When $\theta = 0$, the growth rates g_1 and g_2 are equal. Indeed, using (56) and (57), we have $\widetilde{b}_1 = \widetilde{b}_2$ and, using (22) and (46), the expressions of the growth rates are identical in the YS and AI models.

From (22) and (56), we obtain:

$$g_1 = \frac{\theta s(1-s)A(\beta+\gamma)}{\alpha\phi(1-s)+\theta s(1+\phi)(\alpha+\beta+\gamma)} + \frac{\gamma A}{\beta+\gamma}\left[\frac{\phi}{1+\phi}(1-s)+\theta s\right]$$
$$+\left[\frac{\gamma}{\beta+\gamma} - \theta s\frac{\alpha+\beta+\gamma}{\alpha\frac{\phi}{1+\phi}(1-s)+\theta s(\alpha+\beta+\gamma)}\right]$$
$$\left(\frac{\frac{(1-s)A}{1+\phi}\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\beta+\gamma} - \theta sA}{\frac{\alpha+\pi(\beta+\gamma)}{\alpha+\beta+\gamma} + \frac{\theta s(1-\pi)(1+\phi)}{\phi(1-s)+\theta s(1+\phi)}\frac{\beta+\gamma}{\alpha+\beta+\gamma}}\right).$$

Differentiating this equation, we deduce:

$$\left.\frac{dg_1}{d\theta}\right|_{\theta=0} = sA\left\{\begin{array}{c}\frac{(\beta+\gamma)}{\alpha\phi} + \frac{\gamma}{\beta+\gamma} - \left(\frac{\alpha+\beta+\gamma}{\alpha\phi}\right)\left(\frac{\pi(\beta+\gamma)-\alpha\phi}{\alpha+\pi(\beta+\gamma)}\right)\\ -\frac{\gamma}{\beta+\gamma}\left(\frac{[\alpha+\pi(\beta+\gamma)]+[\pi(\beta+\gamma)-\alpha\phi]\left(\frac{(1-\pi)}{\phi}\frac{\beta+\gamma}{\alpha+\beta+\gamma}\right)}{\frac{[\alpha+\pi(\beta+\gamma)]^2}{\alpha+\beta+\gamma}}\right)\end{array}\right\}.$$

We focus now on g_2. We note that from (44), we obtain $\frac{d\Psi(\widetilde{b}_{2t+1})}{d\widetilde{b}_2}|_{\theta=0} = 0$. Then, using (57), we get:

$$\widetilde{b}_2\big|_{\theta=0} = \frac{(1-s)A}{1+\phi} - \frac{(1-s)A\alpha}{\alpha+(\beta+\gamma)\pi},$$

$$\left.\frac{d\widetilde{b}_2}{d\theta}\right|_{\theta=0} = -\frac{sA(\alpha+\beta+\gamma)}{\alpha+(\beta+\gamma)\pi}.$$

Therefore, using (46), we deduce that:

$$\left.\frac{dg_2}{d\theta}\right|_{\theta=0} = sA\left\{\begin{array}{c}\frac{\gamma}{\beta+\gamma}+\frac{(\beta+\gamma)}{\alpha\phi}-\frac{(\alpha+\beta+\gamma)}{\alpha+(\beta+\gamma)\pi}\left(\frac{\gamma}{\beta+\gamma}\right)\\ -\left(1-\frac{(1+\phi)\alpha}{\alpha+(\beta+\gamma)\pi}\right)\left[\frac{\alpha+(\beta+\gamma)\pi}{\pi\alpha\phi}\right]\end{array}\right\}.$$

We compare now the two derivatives. The inequality $\frac{dg_1}{d\theta}|_{\theta=0} > \frac{dg_2}{d\theta}|_{\theta=0}$ is satisfied if and only if:

$$[\pi(\beta+\gamma)-\alpha\phi] > [\pi(\beta+\gamma)-\alpha\phi]\left(\frac{\gamma\pi}{\alpha+\pi(\beta+\gamma)}\right).$$

A positive bubble at the two BGPs requires $\pi(\beta+\gamma)-\alpha\phi > 0$. Thus, $\frac{dg_1}{d\theta}|_{\theta=0} > \frac{dg_2}{d\theta}|_{\theta=0}$ holds. Since $g_1|_{\theta=0} = g_2|_{\theta=0}$, we conclude that, by continuity, $g_1 > g_2$ for a sufficiently low $\theta > 0$.

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SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Data S1